Conservative Rewritability of Description Logic TBoxes

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Abstract

We investigate the problem of conservative rewritability of a TBox $\mathcal{T}$ in a description logic (DL) $\mathcal{L}$ into a TBox $\mathcal{T}'$ in a weaker DL $\mathcal{L}'$. We focus on model-conservative rewritability ($\mathcal{T}'$ entails $\mathcal{T}$ and all models of $\mathcal{T}$ are expandable to models of $\mathcal{T}'$), subsumption-conservative rewritability ($\mathcal{T}'$ entails $\mathcal{T}$ and all subsumptions in the signature of $\mathcal{T}$ entailed by $\mathcal{T}'$ are entailed by $\mathcal{T}$), and standard DLs between $\mathcal{ALC}$ and $\mathcal{ALCQI}$. We give model-theoretic characterizations of conservative rewritability via bisimulations, inverse p-morphisms and generated subinterpretations, and use them to obtain a few rewriting algorithms and complexity results for deciding rewritability.

1 Introduction

Over the past 30 years, a multitude of description logics (DLs) have been designed, investigated, and used in practice as ontology languages. The introduction of new DLs has been driven by (*i*) the need for additional expressive power (e.g., transitive roles in the 1990s), and (*ii*) applications that require efficient reasoning of a novel type (e.g., ontology-based data access in the 2000s). While the resulting flexibility in choosing DLs has had the positive effect of making DLs available for a large number of domains and applications, it has also led to the development of ontologies with language constructors that are not really required to represent their knowledge. A 'not required' constructor can mean different things here, ranging from the high-level 'this domain can be represented in an adequate way in a weaker DL' to the very concrete 'this ontology is logically equivalent to an ontology in a weaker DL'. In this paper, we take the latter understanding as a starting point. Equivalent rewritability of a DL ontology (TBox) to a weaker language has been investigated by Lutz, Piro, and Wolter [2011] who established model-theoretic characterizations in terms of (various types of) global bisimilations and applied them to the problem of deciding equivalent rewritability. However, equivalent rewritability seems to be an unnecessarily strong condition for multiple applications where *fresh* symbols can be used in rewritings.

Therefore, in this paper, we propose a more flexible notion of *conservative rewritability* that allows the use of fresh symbols in a rewriting $\mathcal{T}'$ of a given TBox $\mathcal{T}$. We demand that $\mathcal{T}'$ entails $\mathcal{T}$. On the other hand, to avoid uncontrolled additional consequences of $\mathcal{T}'$, we also require that (*i*) it does not entail any new subsumptions in the signature of $\mathcal{T}$, or even that (*ii*) every model of $\mathcal{T}$ can be expanded to a model of $\mathcal{T}'$. The latter type of conservative rewriting is known as *model-conservative extension* [Konev et al., 2013], and we call a TBox $\mathcal{T}$ model-conservatively $\mathcal{L}$-rewritable if there is a model-conservative extension of $\mathcal{T}$ in the DL $\mathcal{L}$. The former type is known as a *subsumption* or *deductive conservative extension* [Ghilardi, Lutz, and Wolter, 2006] and, given a DL $\mathcal{L}$, an $\mathcal{L}$ TBox $\mathcal{T}$ and a weaker DL $\mathcal{L}'$, we call $\mathcal{T}$ subsumption-conservatively $\mathcal{L}'$-rewritable if there is a TBox $\mathcal{T}'$ in $\mathcal{L}'$ such that $\mathcal{T}'$ entails the same $\mathcal{L}$-subsumptions in the signature of $\mathcal{T}$ as $\mathcal{T}$. Model-conservative rewritability is a more robust notion as it is language-independent and not only leaves unchanged the entailed subsumptions of the original TBox but also, for example, certain answers in case the ontologies are used to access data.

The main aim of this paper is to show that, for many important DLs, model- and subsumption-conservative rewritabilities can be characterized in terms of natural model-theoretic preservation conditions. In fact, the role played by global bisimilations for equivalent rewritability is now played by generated subinterpretations and p-morphisms (or bounded morphisms), that is, functional bisimilations introduced in modal logic as basic truth-preserving operations on Kripke frames and models [Goranko and Otto, 2006]. We also observe that, in some cases, these characterizations give rise to rewriting algorithms and complexity bounds for deciding conservative rewritability. The latter results are in sharp contrast to the fact that it is typically undecidable whether a given TBox is a model-conservative rewriting of another TBox [Lutz and Wolter, 2010; Konev et al., 2013]. We focus on standard DLs between $\mathcal{ALC}$ and $\mathcal{ALCQI}$, but also briefly consider rewritings into the lightweight DL *DL-Lite*$_{horn}$.

Our model-theoretic characterizations are summarized in Table 1, where the criteria for equivalent rewritability are taken from [Lutz, Piro, and Wolter, 2011]. Thus, for example, model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability coincides with subsumption-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$-rewritability, and both are characterized by preservation under generated subinterpretations or, equivalently, inverse p-morphisms. In contrast, model-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritabil-

Rewritability	Equivalent	Model Conservative	Subsumption Conservative
$\mathcal{ALCI}$-to-$\mathcal{ALC}$	global bisimulations	generated subinterpretations/p-morphisms^{-1}	
$\mathcal{ALCQI}$-to-$\mathcal{ALCQ}$	global counting bisimulations	counting p-morphisms^{-1}	
$\mathcal{ALCQ}$-to-$\mathcal{ALC}$	global bisimulations		p-morphisms^{-1}
$\mathcal{ALCQI}$-to-$\mathcal{ALCI}$	global i-bisimulations	?	i-p-morphisms^{-1}
$\mathcal{ALCQI}$-to-$\mathcal{ALC}$	global bisimulations	?	p-morphisms^{-1}
$\mathcal{ALCI}$-to-*DL-Lite*$_{horn}$	products and succ-simulations		

Table 1: Model-theoretic characterizations of rewritability.

ity coincides with equivalent $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability, but not with subsumption-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability. The situation is yet again different for $\mathcal{ALCI}$-to-*DL-Lite*$_{horn}$ rewritability, in which case all three notions coincide. The question marks indicate two cases where characterizations are unknown.

An in-depth exploration of the applicability of our model-theoretic characterizations is beyond the scope of this paper. We only mention in passing three cases that come naturally along with the preservation criteria. Thus, we show that the preservation conditions for $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability are decidable in ExpTime and give an algorithm constructing polynomial-size rewritings, while those for model-conservative and subsumption-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritabilities give rise to 2ExpTime decision algorithms.

Related work. Conservative rewritings of TBoxes are ubiquitous in DL research. For example, transformations of TBoxes into normal forms are often model-conservative [Baader, Brandt, and Lutz, 2005; Kazakov, 2009]. We note, however, that some well known DL rewritings introducing fresh symbols that are used as a pre-processing step in reasoning [Ding, Haarslev, and Wu, 2007; Carral et al., 2014b; 2014a] or to prove complexity results for reasoning [De Giacomo, 1995] are not conservative rewritings but only satisfiability preserving. There has been significant work on rewritings of ontology-mediated queries (OMQs), which preserve their certain answers, into datalog or OMQs in weaker DLs [Kaminski and Cuenca Grau, 2013; Bienvenu et al., 2014]. It seems that, from a technical viewpoint, rewritability of OMQs is not related to TBox conservative rewritability. Baader [1996] considers the expressive power of DLs and corresponding notions of rewritability based on a variant of model-conservative extension and discusses the relationship to subsumption-conservative extensions. Another closely related problem is TBox approximation. In this case, rather than aiming at a conservative rewriting, the aim is to compute a TBox in a weaker DL that approximates the consequences of the original TBox [Ren, Pan, and Zhao, 2010; Console et al., 2014].

Detailed proofs can be found in [Konev et al., 2016].

2 Conservative Rewritability

In DLs, *concepts* and *roles* are defined inductively starting from countably infinite sets $\mathsf{N_C}$ of *concept names* and $\mathsf{N_R}$ of *role names* and using a set of constructors. The constructors available in $\mathcal{ALCQI}$ are shown in the table below, where the formation of inverse roles is the only role constructor and the remaining four are concept constructors. The third column defines the *extensions* of roles and concepts with these constructors in an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\cdot^{\mathcal{I}}$ maps each concept name A to a subset $A^{\mathcal{I}}$ of the domain $\Delta^{\mathcal{I}}$ of $\mathcal{I}$, and each role name r to $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. In the table, r stands for a role (i.e., a role name or its inverse), A, B for concept names, and C, D for (possibly compound) concepts; $r^{\mathcal{I}}(d) = \{d' \mid (d, d') \in r^{\mathcal{I}}\}$ and $|\Delta|$ is the cardinality of a set Δ. As usual, we define $\top$, $\bot$, $\sqcup$, $\rightarrow$ and $\leftrightarrow$ as standard Boolean abbreviations, $\exists r.C$ (*existential restriction*) as an abbreviation for $(\geqslant 1\ r\ C)$, and $\forall r.C$ (*universal restriction*) for $(\leqslant 0\ r\ \neg C)$. In the sublanguage $\mathcal{ALCQ}$ of $\mathcal{ALCQI}$, inverse roles are disallowed; in $\mathcal{ALCI}$, at-least and at-most restrictions are limited to $\exists r.C$ and $\forall r.C$; and $\mathcal{ALC}$ is the common part of $\mathcal{ALCQ}$ and $\mathcal{ALCI}$.

constructor	syntax	semantics
inverse role	r^-	$(r^{\mathcal{I}})^{-1} = \{(d,e) \mid (e,d) \in r^{\mathcal{I}}\}$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
at-least restriction	$(\geqslant n\ r\ C)$	$\{d \in \Delta^{\mathcal{I}} \mid \lvert r^{\mathcal{I}}(d) \cap C^{\mathcal{I}}\rvert \geq n\}$
at-most restriction	$(\leqslant n\ r\ C)$	$\{d \in \Delta^{\mathcal{I}} \mid \lvert r^{\mathcal{I}}(d) \cap C^{\mathcal{I}}\rvert \leq n\}$

An $\mathcal{L}$ *TBox*, $\mathcal{T}$, for a DL $\mathcal{L}$ is a finite set of *concept inclusions* (CI) of the form $C \sqsubseteq D$, where C and D are $\mathcal{L}$-concepts. We write $\mathcal{I} \models C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $\mathcal{I} \models \mathcal{T}$ if this holds for all CIs in $\mathcal{T}$, in which case $\mathcal{I}$ is said to be a *model* of $\mathcal{T}$. $\mathcal{T}$ is *consistent* if it has a model. By a *signature*, Σ, we mean any set of concept and role names. The *signature* $\mathsf{sig}(\mathcal{T})$ *of a TBox* $\mathcal{T}$ is the set of concept and role names occurring in $\mathcal{T}$. By $\mathsf{sub}(\mathcal{T})$ we denote the closure under single negation of the set of subconcepts of concepts in $\mathcal{T}$.

Now we define three notions of TBox rewritability for DLs $\mathcal{L}$ and $\mathcal{L}'$, where $\mathcal{L}$ is typically more expressive than $\mathcal{L}'$.

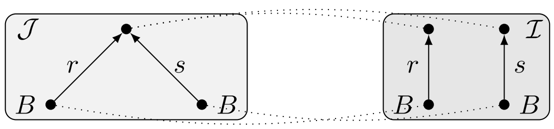
Definition 1 An $\mathcal{L}'$ TBox $\mathcal{T}'$ is an *equivalent $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}$ and $\mathcal{T}'$ have the same models. $\mathcal{T}$ is *equivalently $\mathcal{L}'$-rewritable* if it has an equivalent $\mathcal{L}'$-rewriting.

Equivalent $\mathcal{L}$-to-$\mathcal{L}'$ rewritability was studied by Lutz, Piro, and Wolter [2011], who gave the semantic characterizations in the first column of Table 1. For example, if $\mathcal{L}$ is $\mathcal{ALCQI}$, $\mathcal{ALCI}$ or $\mathcal{ALCQ}$ and $\mathcal{L}'$ is $\mathcal{ALC}$, then an $\mathcal{L}$ TBox is equivalently $\mathcal{L}'$-rewritable iff its class of models is preserved under *global bisimulations*, which are defined as follows. Given a signature Σ, a Σ-*bisimulation* between interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ is a relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ that satisfies conditions

[Atom], [Forth] and [Back] in the table below, for $r, A \in \Sigma$. In [Back] and elsewhere, 'dual' refers to swapping the rôles of $\mathcal{I}_1, d_1, d_1'$ and $\mathcal{I}_2, d_2, d_2'$. The relation S is a *global* Σ-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ if $\Delta^{\mathcal{I}_1}$ is the domain of S and $\Delta^{\mathcal{I}_2}$ its range. $\mathcal{I}_1$ and $\mathcal{I}_2$ are *globally* Σ*-bisimilar* if there is a global Σ-bisimulation between them. If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, we omit Σ and say simply '(global) bisimulation.' A TBox $\mathcal{T}$ is *preserved under global bisimulations* if any interpretation that is globally bisimilar to a model of $\mathcal{T}$ is a model of $\mathcal{T}$.

[Atom]	for all $(d_1, d_2) \in S$, $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$
[Forth]	if $(d_1, d_2) \in S$ and $d_1' \in r^{\mathcal{I}_1}(d_1)$, $r \in \mathsf{N_R}$, then there is a $d_2' \in r^{\mathcal{I}_2}(d_2)$ with $(d_1', d_2') \in S$.
[Back]	dual of [Forth]
[QForth]	if $(d_1, d_2) \in S$ and $D_1 \subseteq r^{\mathcal{I}_1}(d_1)$ is finite, $r \in \mathsf{N_R}$, then there is a $D_2 \subseteq r^{\mathcal{I}_2}(d_2)$ such that S contains a bijection between D_1 and D_2.
[QBack]	dual of [QForth]

Example 1 The $\mathcal{ALCI}$ TBox $\{\exists r^-.B \sqsubseteq A\}$ can be equivalently rewritten to the $\mathcal{ALC}$ TBox $\{B \sqsubseteq \forall r.A\}$. However, the $\mathcal{ALCI}$ TBox $\mathcal{T} = \{\exists r^-.B \sqcap \exists s^-.B \sqsubseteq A\}$ is not equivalently $\mathcal{ALC}$-rewritable. Indeed, the interpretation $\mathcal{I}$ in the picture below is a model of $\mathcal{T}$ and globally bisimilar to the interpretation $\mathcal{J}$, which is not a model of $\mathcal{T}$.



Equivalent $\mathcal{ALCQI}$-to-$\mathcal{ALCQ}$ rewritability is characterized by *counting bisimulations* defined but replacing [Forth] and [Back] in the definition of bisimulations with [QForth] and [QBack]. For equivalent $\mathcal{ALCQI}$-to-$\mathcal{ALCI}$ rewritability, we need i*-bisimulations*, that is, bisimulations for which [Forth] and [Back] hold for inverse roles as well.

We now introduce two subtler notions of TBox rewritability, which allow the use of fresh concept and role names in rewritings. For an interpretation $\mathcal{I}$ and a signature Σ, the Σ*-reduct of* $\mathcal{I}$ is the interpretation $\mathcal{I}_{|\Sigma}$ coinciding with $\mathcal{I}$ on Σ and having $X^{\mathcal{I}_{|\Sigma}} = \emptyset$ for $X \notin \Sigma$. We say that interpretations $\mathcal{I}$ and $\mathcal{J}$ *coincide on* Σ and write $\mathcal{I} =_\Sigma \mathcal{J}$ if the Σ-reducts of $\mathcal{I}$ and $\mathcal{J}$ coincide. A TBox $\mathcal{T}'$ is called a *model-conservative* (or m-conservative) *extension* of $\mathcal{T}$ if $\mathcal{T}' \models \mathcal{T}$ and, for every $\mathcal{I} \models \mathcal{T}$, there is $\mathcal{I}' \models \mathcal{T}'$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

Definition 2 An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called an m*-conservative* $\mathcal{L}'$*-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}'$ is an m-conservative extension of $\mathcal{T}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is m*-conservatively* $\mathcal{L}'$*-rewritable* if it has an m-conservative $\mathcal{L}'$-rewriting.

Any equivalent $\mathcal{L}'$-rewriting of a TBox $\mathcal{T}$ is also an m-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$, but not the other way round:

Example 2 The $\mathcal{ALCI}$ TBox $\{\exists r^-.B \sqcap \exists s^-.B \sqsubseteq A\}$ from Example 1 is m-conservatively $\mathcal{ALC}$-rewritable to

$$\{B \sqsubseteq \forall r.B_{\exists r^-.B}, B \sqsubseteq \forall s.B_{\exists s^-.B}, B_{\exists r^-.B} \sqcap B_{\exists s^-.B} \sqsubseteq A\},$$

where $B_{\exists r^-.B}$, $B_{\exists s^-.B}$ are *fresh* concept names.

A TBox $\mathcal{T}'$ is a *subsumption-conservative* (s-conservative) *extension* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}' \models \mathcal{T}$ and $\mathcal{T}' \models C \sqsubseteq D$ implies $\mathcal{T} \models C \sqsubseteq D$, for any $\mathcal{L}$-CI $C \sqsubseteq D$ given in $\mathsf{sig}(\mathcal{T})$.

Definition 3 An $\mathcal{L}'$ TBox $\mathcal{T}'$ is an s*-conservative* $\mathcal{L}'$*-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}'$ is an s-conservative extension of $\mathcal{T}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is s*-conservatively* $\mathcal{L}'$*-rewritable* if it has an s-conservative $\mathcal{L}'$-rewriting.

Note that it makes sense to speak about an s-conservative $\mathcal{L}'$-rewriting of a TBox $\mathcal{T}$ only if the language of $\mathcal{T}$ is understood. For example, the $\mathcal{ALC}$ TBox $\{\top \sqsubseteq \exists r.A \sqcap \exists r.\neg A\}$ is an s-conservative rewriting of $\mathcal{T} = \{\top \sqsubseteq \exists r.\top\}$ when $\mathcal{T}$ is regarded as an $\mathcal{ALC}$ TBox, but not as an $\mathcal{ALCQ}$ TBox.

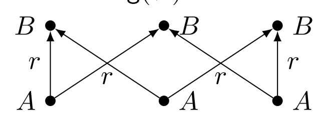
Every m-conservatively $\mathcal{L}'$-rewritable TBox $\mathcal{T}$ is s-conservatively $\mathcal{L}'$-rewritable, but not the converse:

Example 3 The $\mathcal{ALCQ}$ TBox $\mathcal{T} = \{A \sqsubseteq\ \geq 2\, r.B\}$ is s-conservatively $\mathcal{ALC}$-rewritable to

$$\mathcal{T}' = \{A \sqsubseteq \exists r.B_1,\ A \sqsubseteq \exists r.B_2,\ B_1 \sqsubseteq \neg B_2,\ B_1 \sqcup B_2 \sqsubseteq B\},$$

where B_1 and B_2 are *fresh* concept names. To show this, note first that $\mathcal{T}' \models \mathcal{T}$. Second, recall that $\mathcal{ALCQ}$ TBoxes are complete for *ditree interpretations*, that is, interpretations $\mathcal{I}$ such that $r^\mathcal{I} \cap s^\mathcal{I} = \emptyset$ for $r \neq s$ and the directed graph with nodes $\Delta^\mathcal{I}$ and edges $(d, d') \in \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}$ is a directed tree. Thus, if $\mathcal{T} \not\models C \sqsubseteq D$, for an $\mathcal{ALCQ}$-CI $C \sqsubseteq D$ in $\mathsf{sig}(\mathcal{T})$, then there is a ditree model $\mathcal{I}$ of $\mathcal{T}$ with $\mathcal{I} \not\models C \sqsubseteq D$. Clearly, there exists a model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. But then $\mathcal{J} \not\models C \sqsubseteq D$, and so $\mathcal{T}' \not\models C \sqsubseteq D$, as required.

However, $\mathcal{T}'$ is not an m-conservative rewriting of $\mathcal{T}$ because (in contrast to ditree models of $\mathcal{T}$) the model $\mathcal{I}$ of $\mathcal{T}$ shown below is not the $\mathsf{sig}(\mathcal{T})$-reduct of any model of $\mathcal{T}'$.



It is not difficult to generalize this argument to prove that there is *no* m-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$.

In our examples so far, we used fresh concept names but no fresh role names. This is no accident: for the DLs considered in this paper, fresh role names in conservative rewritings are not required. Say that a DL $\mathcal{L}$ *reflects disjoint unions* if, for any $\mathcal{L}$ TBox $\mathcal{T}$, whenever the *disjoint* union $\bigcup_{i \in I} \mathcal{I}_i$ of interpretations $\mathcal{I}_i$ is a model of $\mathcal{T}$, then each $\mathcal{I}_i$, $i \in I$, is a model of $\mathcal{T}$. All of our DLs are known to reflect disjoint unions.

Theorem 1 *Let $\mathcal{L}$ be a DL reflecting disjoint unions, $\mathcal{T}$ an $\mathcal{L}$ TBox, and let $\mathcal{L}' \in \{\mathcal{ALCQ}, \mathcal{ALCI}, \mathcal{ALC}\}$. If $\mathcal{T}$ is* m*-conservatively* (*or* s*-conservatively*) *$\mathcal{L}'$-rewritable, then $\mathcal{T}$ has a* m*-conservative* (*or, respectively,* s*-conservative*) *$\mathcal{L}'$-rewriting not using role names outside* $\mathsf{sig}(\mathcal{T})$.

Proof. To illustrate the idea, consider an m-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$ of $\mathcal{T}$. For any $C \in \mathsf{sub}(\mathcal{T}')$ of the form $\exists r.C'$ or $\forall r.C'$ with $r \notin \mathsf{sig}(\mathcal{T})$, take a fresh concept name B_C and denote by $D^\sharp$ the result of replacing all top-most occurrences of such C in $D \in \mathsf{sub}(\mathcal{T}')$ by B_C. The required m-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}^\dagger$ is given by the inclusions $\bigsqcap_{C \in \boldsymbol{t}} C^\sharp \sqsubseteq \bot$, where $\boldsymbol{t}$ ranges over maximal subsets of $\mathsf{sub}(\mathcal{T})$ such that $\bigsqcap_{C \in \boldsymbol{t}} C$ is not satisfiable with respect to $\mathcal{T}'$. Indeed, for any $\mathcal{I} \models \mathcal{T}$, there is $\mathcal{J} \models \mathcal{T}^\dagger$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. To show $\mathcal{T}^\dagger \models \mathcal{T}$, suppose $\mathcal{I} \models \mathcal{T}^\dagger$ and $\mathcal{I} \not\models \mathcal{T}$, with $r^\mathcal{I} = \emptyset$ for $r \notin \mathsf{sig}(\mathcal{T})$. By the definition of $\mathcal{T}^\dagger$, for every $d \in \Delta^\mathcal{I}$, there is a ditree model $\mathcal{I}_d$ of $\mathcal{T}'$ with root d (and no other shared elements) such that $d \in (C^\sharp)^\mathcal{I}$ iff $d \in C^{\mathcal{I}_d}$, for $C \in \mathsf{sub}(\mathcal{T}')$. We remove all $(d,d') \in r^{\mathcal{I}_d}$ with $r \in \mathsf{sig}(\mathcal{T})$ from $\mathcal{I}_d$, $d \in \Delta^\mathcal{I}$, and take the union $\mathcal{J}$ of the resulting interpretations with $\mathcal{I}$. Then $\mathcal{J} \models \mathcal{T}'$ but $\mathcal{J} \not\models \mathcal{T}$ (because $\mathcal{T}$ reflects disjoint unions and $\mathcal{J}_{|\mathsf{sig}(\mathcal{T})}$ is the disjoint union of the $\mathsf{sig}(\mathcal{T})$-reduct of $\mathcal{I}$ and the $\mathsf{sig}(\mathcal{T})$-reducts of $\mathcal{I}_d$ with d removed), which is a contradiction. ❑

Note that the size of $\mathcal{T}^\dagger$ is exponential in $|\mathcal{T}|$. It is an interesting open problem whether a polynomial rewriting exists. To see why reflection of disjoint unions is essential, consider the $\mathcal{ALCU}$ TBox $\mathcal{T} = \{\top \sqsubseteq \exists u.A\}$ with the *universal role* u, which is a logical symbol and not part of the signature of $\mathcal{T}$ [Krötzsch, Simančík, and Horrocks, 2012]. Then $\{\top \sqsubseteq \exists r.A\}$ is an m-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$ but no such rewriting without fresh role names exists.

3 Rewriting Inverse Roles

In this section, we investigate conservative TBox rewritability from DLs with inverse roles to the corresponding DLs without them. First, we give a natural characterization of m- and s-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCI}$-TBoxes in terms of generated subinterpretations. Motivated by the observation that preservation under generated subinterpretations does *not* characterize conservative $\mathcal{ALCQI}$-to-$\mathcal{ALCQ}$ rewritability, we then give an alternative characterization of conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability in terms of p-morphisms. In contrast to generated subinterpretations, p-morphisms can be lifted to $\mathcal{ALCQI}$, and we show that m- and s-conservative $\mathcal{ALCQI}$-to-$\mathcal{ALCQ}$ rewritability is characterized in terms of counting p-morphisms.

An interpretation $\mathcal{I}$ is a *subinterpretation* of $\mathcal{J}$ if $\Delta^\mathcal{I} \subseteq \Delta^\mathcal{J}$, $A^\mathcal{I} = A^\mathcal{J} \cap \Delta^\mathcal{I}$, and $r^\mathcal{I} = r^\mathcal{J} \cap (\Delta^\mathcal{I} \times \Delta^\mathcal{I})$ for all A and r. $\mathcal{I}$ is a *generated subinterpretation* of $\mathcal{J}$ if, in addition, $d \in \Delta^\mathcal{I}$ and $(d,d') \in r^\mathcal{J}$ imply $d' \in \Delta^\mathcal{I}$. A TBox $\mathcal{T}$ is *preserved under generated subinterpretations* if every generated subinterpretation of a model of $\mathcal{T}$ is also a model of $\mathcal{T}$. As well known, all $\mathcal{ALC}$ TBoxes enjoy this property.

Suppose we want to construct an m-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCI}$ TBox $\mathcal{T}$. Without loss of generality, we can assume that $\mathcal{T}$ uses the concept constructors $\neg$, $\sqcap$ and $\exists$ only. For any role name r in $\mathcal{T}$, take a fresh role name $\bar{r}$. Then, for any $\exists r.C$ in $\mathsf{sub}(\mathcal{T})$, where r is a role (a role name or its inverse), take a fresh concept name $B_{\exists r.C}$. Denote by $D^\sharp$ the $\mathcal{ALC}$-concept obtained from any $D \in \mathsf{sub}(\mathcal{T})$ by replacing every top-most occurrence of a subconcept of the form $\exists r.C$ in D with $B_{\exists r.C}$. Now, let $\mathcal{T}^\dagger$ be an $\mathcal{ALC}$ TBox containing $C^\sharp \sqsubseteq D^\sharp$, for $C \sqsubseteq D \in \mathcal{T}$, and for $r \in \mathsf{N_R}$,

$$C^\sharp \sqsubseteq \forall \bar{r}.B_{\exists r.C}, \quad B_{\exists r.C} \equiv \exists r.C^\sharp, \quad \text{for } \exists r.C \in \mathsf{sub}(\mathcal{T}),$$

$$C^\sharp \sqsubseteq \forall r.B_{\exists r^-.C}, \; B_{\exists r^-.C} \equiv \exists \bar{r}.C^\sharp, \; \text{for } \exists r^-.C \in \mathsf{sub}(\mathcal{T}).$$

Clearly, $\mathcal{T}^\dagger$ can be constructed in polynomial time in $|\mathcal{T}|$.

Theorem 2 *The following conditions are equivalent for any $\mathcal{ALCI}$ TBox $\mathcal{T}$:*

(1) *$\mathcal{T}$ is m-conservatively $\mathcal{ALC}$-rewritable;*

(2) *$\mathcal{T}$ is s-conservatively $\mathcal{ALC}$-rewritable;*

(3) *$\mathcal{T}$ is preserved under generated subinterpretations;*

(4) *$\mathcal{T}^\dagger$ is an m-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$.*

Proof. We only briefly discuss the proof of (3) $\Rightarrow$ (4) here. Assume (3). Clearly, for every model $\mathcal{I}$ of $\mathcal{T}$, there is a model $\mathcal{J}$ of $\mathcal{T}^\dagger$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. It remains to show that $\mathcal{T}^\dagger \models \mathcal{T}$. Suppose $\mathcal{I} \models \mathcal{T}^\dagger$. The extension $\mathcal{I}_1$ of $\mathcal{I}$ in which the interpretation of every $\bar{r}$ is extended by the inverse of $r^\mathcal{I}$ is also a model of $\mathcal{T}^\dagger$. Let $\mathcal{I}_2$ be $\mathcal{I}_1$ with every $d \in \Delta^{\mathcal{I}_1}$ renamed to d'. Take the disjoint union $\mathcal{J}$ of $\mathcal{I}_1$ and $\mathcal{I}_2$, and replace each $(d,e) \in \bar{r}^\mathcal{J}$ such that $d, e \in \Delta^{\mathcal{I}_1}$ and $(e,d) \notin r^\mathcal{J}$ with $(e',d) \in r^\mathcal{J}$ and $(d,e') \in \bar{r}^\mathcal{J}$, and add $(e',d') \in r^\mathcal{J}$ for any $(d',e') \in \bar{r}^\mathcal{J}$ with $d', e' \in \Delta^{\mathcal{I}_2}$ and $(e',d') \notin r^\mathcal{J}$. Then $\mathcal{J} \models \mathcal{T}$, with the $\mathsf{sig}(\mathcal{T})$-reduct of $\mathcal{I}$ being a generated subinterpretation of the $\mathsf{sig}(\mathcal{T})$-reduct of $\mathcal{J}$. Thus $\mathcal{I} \models \mathcal{T}$. ❑

It is open whether a polynomial-size rewriting *without* additional role names exists. The proof above shows that to decide whether $\mathcal{T}$ is m-conservatively $\mathcal{ALC}$-rewritable, it is enough to check whether $\mathcal{T}^\dagger \models \mathcal{T}$, which can be done in ExpTime [Baader et al., 2003]. A matching ExpTime lower bound is obtained by reduction of $\mathcal{ALCI}$ TBox satisfiability.

Corollary 1 *Deciding m-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is ExpTime-complete.*

The next example shows that preservation under generated subinterpretations does not guarantee conservative $\mathcal{ALCQI}$-to-$\mathcal{ALCQ}$ rewritability.

Example 4 Any subinterpretation of a model of the $\mathcal{ALCQI}$ TBox $\mathcal{T} = \{A \sqsubseteq (\leq 1\; r^-.\top)\}$ is also a model of $\mathcal{T}$, and so $\mathcal{T}$ is preserved under generated subinterpretations. We prove below that $\mathcal{T}$ is not m-conservatively $\mathcal{ALCQ}$ rewritable.

The reason why $\mathcal{T}$ cannot be conservatively rewritten into an $\mathcal{ALCQ}$ TBox is that, without inverse roles, one cannot restrict the number of r-predecessors. To capture this intuition, we introduce a functional version of (counting) bisimulations.

Definition 4 A (*counting*) *Σ-p-morphism* from $\mathcal{I}_1$ to $\mathcal{I}_2$ is any global (counting) Σ-bisimulation S between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that S is a function. If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, we refer to S as a (counting) p-morphism. A TBox $\mathcal{T}$ is *preserved under inverse* (*counting*) *p-morphisms* if $\mathcal{I} \models \mathcal{T}$ whenever there is a (counting) p-morphism from $\mathcal{I}$ to a model of $\mathcal{T}$.

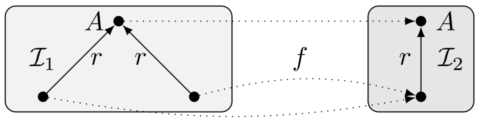
A fundamental property of p-morphisms is established by

Lemma 1 *Suppose $\mathcal{T}$ is an $\mathcal{ALC}$ (or $\mathcal{ALCQ}$) TBox, Σ contains all role names in $\mathsf{sig}(\mathcal{T})$, and there is a (counting) Σ-p-morphism f from an interpretation $\mathcal{I}$ to some model $\mathcal{I}'$ of $\mathcal{T}$. Then there is a model $\mathcal{J}$ of $\mathcal{T}$ such that $\mathcal{J} =_\Sigma \mathcal{I}$.*

Proof. We define $\mathcal{J}$ in the same way as $\mathcal{I}$ except that we set $A^\mathcal{J} = f^{-1}(A^{\mathcal{I}'})$ for $A \in \mathsf{sig}(\mathcal{T}) \setminus \Sigma$. Then f is a (counting) $\mathsf{sig}(\mathcal{T})$-bisimulation from $\mathcal{J}$ to $\mathcal{I}'$, and so $\mathcal{J} \models \mathcal{T}$. ❑

It follows that if an $\mathcal{ALCI}$ (or $\mathcal{ALCQI}$) TBox $\mathcal{T}$ is m-conservatively $\mathcal{ALC}$- (or $\mathcal{ALCQ}$-) rewritable, then $\mathcal{T}$ is preserved under inverse (counting) p-morphisms. Indeed, let $f : \mathcal{I}_1 \to \mathcal{I}_2$ be a p-morphism and $\mathcal{T}'$ an m-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$. By Theorem 1, we may assume that the role names in $\mathsf{sig}(\mathcal{T}')$ belong to $\mathsf{sig}(\mathcal{T})$. By Lemma 1, there is a model $\mathcal{J}_1$ of $\mathcal{T}'$ with $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1$, from which $\mathcal{I}_1 \models \mathcal{T}$.

Example 5 The map $f\colon \mathcal{I}_1 \to \mathcal{I}_2$ below is a counting p-morphism. Since $\mathcal{I}_2$ is a model of $\mathcal{T}$ from Example 4 but $\mathcal{I}_1$ is not, $\mathcal{T}$ is not m-conservatively $\mathcal{ALCQ}$-rewritable.



Note that if a TBox $\mathcal{T}$ reflects disjoint unions and is preserved under inverse p-morphisms, then it is preserved under generated subinterpretations. Indeed, let $\mathcal{I}$ be a generated subinterpretation of $\mathcal{J} \models \mathcal{T}$. Take the disjoint union $\mathcal{I}' = (\mathcal{I} \times \{0\}) \cup (\mathcal{J} \times \{1\})$ of $\mathcal{I}$ and $\mathcal{J}$. The map $f\colon \mathcal{I}' \to \mathcal{J}$ defined by setting $f(d,i) = d$ for $i = 0,1$ is a p-morphism. Then $\mathcal{I}' \models \mathcal{T}$, and so $\mathcal{I} \models \mathcal{T}$. Thus, we obtain:

Theorem 3 *An $\mathcal{ALCI}$ TBox is* m*-conservatively (and* s*-conservatively) $\mathcal{ALC}$-rewritable iff it is preserved under inverse p-morphisms.*

Counting p-morphisms characterize both m- and s-conservative $\mathcal{ALCQ}$-rewritabilities:

Theorem 4 *The following conditions are equivalent for any $\mathcal{ALCQI}$ TBox $\mathcal{T}$:*

(1) *$\mathcal{T}$ is* m*-conservatively $\mathcal{ALCQ}$-rewritable;*

(2) *$\mathcal{T}$ is* s*-conservatively $\mathcal{ALCQ}$-rewritable;*

(3) *$\mathcal{T}$ is preserved under inverse counting p-morphisms.*

Proof. We sketch the proof of (3) $\Rightarrow$ (1) where, unlike Theorem 2, we construct an *infinite* rewriting $\mathcal{T}'$ from which a finite one is obtained by compactness. $\mathcal{T}'$ is defined by brute force: given $\mathcal{T}$, it includes all $C^\sharp \sqsubseteq D^\sharp$ with $\mathcal{T} \models C \sqsubseteq D$, where C, D are $\mathcal{ALCQI}$ concepts over $\mathsf{sig}(\mathcal{T})$ and $C^\sharp, D^\sharp$ are the results of replacing uniformly any top-most qualified number restriction with inverse role by a fresh concept name. The crucial step now is to prove that $\mathcal{T}' \models \mathcal{T}$ if $\mathcal{T}$ is preserved under inverse counting morphisms. Suppose this is not so. Take an ω-saturated model $\mathcal{I}$ of $\mathcal{T}'$ that is not a model of $\mathcal{T}$ [Chang and Keisler, 1990, p. 100]. By unraveling $\mathcal{I}$ into a tree-shaped interpretation and using preservation under inverse counting p-morphisms, we construct a new $\mathcal{I}'$ with $\mathcal{I}' \models \mathcal{T}'$ and $\mathcal{I}' \not\models \mathcal{T}$, in which no node has more than one r-predecessor (r a role name) satisfying the same $\mathcal{ALCQ}$-concepts; cf. Example 5. Now we construct a model $\mathcal{J}$ of $\mathcal{T}$ containing $\mathcal{I}'$ as a generated subinterpretation, contrary to $\mathcal{T}$ being preserved under inverse counting p-morphisms. ❑

The decidability of rewritability and the size of rewritings in Theorem 4 remain open.

4 Rewriting Number Restrictions

Now we consider TBox rewritability from DLs with qualified number restrictions to the corresponding DLs without them. We first characterize s-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability and $\mathcal{ALCQI}$-to-$\mathcal{ALCI}$ rewritability in terms of p-morphisms and, respectively, i-p-morphisms. We then generalize Example 3 and show that m-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability coincides with equivalent $\mathcal{ALC}$-rewritability by characterizing it in terms of preservation under global bisimulations. Finally, we show that this is not the case for m-conservative $\mathcal{ALCQI}$-to-$\mathcal{ALCI}$ rewritability.

The next lemma shows that s-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability can be regarded as a principled approximation of m-conservative rewritability (cf. Example 3).

Lemma 2 *An $\mathcal{ALC}$ TBox $\mathcal{T}'$ is an* s*-conservative rewriting of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ iff $\mathcal{T}'$ is an* m*-conservative rewriting of $\mathcal{T}$ over ditree interpretations of finite outdegree.*

Suppose we need an s-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCQ}$-TBox $\mathcal{T}$. As before, we assume that $\mathcal{T}$ is built using $\neg$, $\sqcap$ and $(\geqslant n\ r\ C)$ only. Take fresh concept names $B_D, B_1^D, \dots, B_n^D$, for $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$, and let Σ be $\mathsf{sig}(\mathcal{T})$ together with the fresh concept names. For $C \in \mathsf{sub}(\mathcal{T})$, let $C^\sharp$ be the $\mathcal{ALC}$-concept obtained from C by replacing all top-most occurrences of $D = (\geqslant n\ r\ D')$ in C with B_D. Let $\mathcal{T}^\dagger$ be the *infinite* TBox containing $C^\sharp \sqsubseteq D^\sharp$, for $C \sqsubseteq D \in \mathcal{T}$, and for $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$,

– $B_i^D \sqsubseteq \neg B_j^D$ for $i \neq j$,

– $B_D \sqsubseteq \exists r.(C^\sharp \sqcap B_1^D) \sqcap \cdots \sqcap \exists r.(C^\sharp \sqcap B_n^D)$,

– $\bigsqcap_{1 \leq i \leq n} (\exists r.(C^\sharp \sqcap C_i^\sharp \sqcap \bigsqcap_{j \neq i} \neg C_j^\sharp)) \sqsubseteq B_D$, for any $\mathcal{ALC}$-concepts C_i with $\mathsf{sig}(C_i) \subseteq \Sigma$.

Theorem 5 *The following conditions are equivalent for any $\mathcal{ALCQ}$ TBox $\mathcal{T}$:*

(1) *$\mathcal{T}$ is* s*-conservatively $\mathcal{ALC}$-rewritable;*

(2) *$\mathcal{T}^\dagger$ is an* s*-conservative (infinite) $\mathcal{ALC}$-rewriting of $\mathcal{T}$;*

(3) *$\mathcal{T}$ is preserved under inverse p-morphisms.*

Proof. We sketch (3) $\Rightarrow$ (2). The interesting step is to prove that $\mathcal{T}^\dagger \models \mathcal{T}$. Suppose this is not the case. We find an ω-saturated model $\mathcal{I}$ of $\mathcal{T}^\dagger$ such that $\mathcal{I} \not\models \mathcal{T}$. Let $\mathcal{J}$ be the quotient $\mathcal{I}/_\sim$, where $d \sim d'$ if (d,d') is contained in the largest Σ-bisimulation on $\mathcal{I}$. The map that sends each $d \in \Delta^\mathcal{I}$ to its equivalence class $d/_\sim$ in $\mathcal{J}$ is a Σ-p-morphism, and by carefully analysing $\mathcal{T}^\dagger$ one can show that $\mathcal{J} \models \mathcal{T}$. By (2), $\mathcal{I} \models \mathcal{T}$, which is a contradiction. ❑

Although we do not know how to decide preservation under inverse counting p-morphisms from Theorem 4, preservation under inverse p-morphisms of $\mathcal{ALCQ}$ TBoxes can be decided in 2ExpTime (with numbers coded in unary). The algorithm uses a type elimination argument similar to the one employed for deciding equivalent $\mathcal{ALC}$-rewritability of $\mathcal{ALCI}$ TBoxes [Lutz, Piro, and Wolter, 2011]. So we have:

Theorem 6 *The problem of* s*-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCQ}$ TBoxes is decidable in* 2ExpTime.

Thus, given an $\mathcal{ALCQ}$ TBox $\mathcal{T}$, one can first decide s-conservative $\mathcal{ALC}$-rewritability and then, in case of a positive answer, effectively construct a rewriting by going through the finite subsets of $\mathcal{T}^\dagger$ in a systematic way until a finite $\mathcal{T}' \subseteq \mathcal{T}^\dagger$ with $\mathcal{T}' \models \mathcal{T}$ is found, which must exist by compactness.

Our analysis of s-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCQ}$ TBoxes can be lifted to s-conservative $\mathcal{ALCI}$-rewritability of $\mathcal{ALCQI}$ TBoxes by replacing (*i*) ditree interpretations with *tree interpretations* (in which $r^\mathcal{I} \cap s^\mathcal{I} = \emptyset$ for all *roles* $r \neq s$, and the undirected graph with nodes $\Delta^\mathcal{I}$ and edges $\{d,d'\}$ for $(d,d') \in \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}$ is a tree); (*ii*) p-morphisms with i-p-morphisms (functional i-bisimulations); and (*iii*) using fresh concept names B_D for qualified number restrictions D with inverse roles as well. These modifications give the required generalizations of Lemma 2 and Theorem 5. However, decidability of s-conservative $\mathcal{ALCI}$-rewritability of $\mathcal{ALCQI}$ TBoxes remains open.

As to m-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability, Example 3 shows that the straightforward s-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$ of $\mathcal{T} = \{A \sqsubseteq \geq 2\, r.B\}$ is not an m-conservative rewriting because there is a *non*-tree interpretation $\mathcal{I}$ for which no $\mathcal{J} \models \mathcal{T}'$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$ exists. A generalization of this argument shows that only $\mathcal{ALCQ}$ TBoxes that are preserved under global bisimulations are m-conservatively $\mathcal{ALC}$-rewritable. Thus, we obain:

Theorem 7 *An $\mathcal{ALCQ}$ TBox is* m*-conservatively $\mathcal{ALC}$-rewritable iff it is equivalently $\mathcal{ALC}$-rewritable.*

Using type elimination, one can prove that deciding preservation of $\mathcal{ALCQ}$ TBoxes under global bisimulations is in 2EXPTIME. Thus, m-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCQ}$ TBoxes is decidable in 2EXPTIME.

Surprisingly, the situation is different for m-conservative $\mathcal{ALCQI}$-to-$\mathcal{ALCI}$ rewritability, where one would also expect that only equivalently $\mathcal{ALCI}$-rewritable TBoxes (those that are preserved under global i-bisimulations) are m-conservatively $\mathcal{ALCI}$-rewritable. However, the following example shows that this is not the case:

Example 6 The TBox $\mathcal{T} = \{\exists r.\top \sqsubseteq \exists r.(\geq\ 2r^-.\top)\}$ in $\mathcal{ALCQI}$ has the m-conservative $\mathcal{ALCI}$-rewriting $\mathcal{T}' = \{\exists r.\top \sqsubseteq \exists r.(\exists r^-.B \sqcap \exists r^-.\neg B)\}$. No equivalent $\mathcal{ALCI}$-rewriting of $\mathcal{T}$ exists because it is not preserved under global i-bisimulations. The proof that, for every $\mathcal{I} \models \mathcal{T}$, there is $\mathcal{J} \models \mathcal{T}'$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$ relies on the observation that non-tree shaped counterexamples such as the one in Example 3 do not exist because of the interaction between $\mathcal{T}$'s constraints for r-successors and r^--successors.

We do not have any conjecture as to a natural semantic characterization of m-conservative $\mathcal{ALCQI}$-to-$\mathcal{ALCI}$ rewritability. In fact, Theorem 7 and Example 6 together suggest that such a characterization does not exist.

5 $\mathcal{ALCQI}$-to-$\mathcal{ALC}$ Rewritability

At first sight, $\mathcal{ALCQI}$-to-$\mathcal{ALC}$ rewritability easily reduces to the two-step $\mathcal{ALCQI}$-to-$\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability. Note, however, that the first step introduces fresh concept names that are not regarded as auxiliary in the second step. In fact, to smoothly compose the two steps, a more general notion of rewritability with a distinguished set of symbols in the input TBox is needed. Call a TBox $\mathcal{T}$ m*-conservatively $\mathcal{L}$ rewritable relative to a signature* $\Sigma \subseteq \mathsf{sig}(\mathcal{T})$ if there exists an $\mathcal{L}$-TBox $\mathcal{T}'$ such that $\{\mathcal{I}_{|\Sigma} \mid \mathcal{I} \models \mathcal{T}\} = \{\mathcal{I}_{|\Sigma} \mid \mathcal{I} \models \mathcal{T}'\}$. Investigating this notion is beyond the scope of this paper. We only mention one unexpected result, which can be proved by reduction of the undecidable problem whether an $\mathcal{ALC}$ TBox is an m-conservative rewriting of the empty TBox [Konev et al., 2013] (cf. Corollary 1):

Theorem 8 *The problem of* m*-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability relative to a signature Σ is undecidable.*

As $\mathcal{ALC}$-rewritable $\mathcal{ALCQI}$ TBoxes are not preserved under global bisimulations (see Example 2), we cannot simply put together the corresponding characterizations from the previous two sections in order to characterize m-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCQI}$ TBoxes. Nevertheless, by applying the s-conservative $\mathcal{ALCQ}$ rewriting above to the rewriting in the proof of Theorem 4, we obtain an s-conservative $\mathcal{ALC}$-rewriting of an input $\mathcal{ALCQI}$ TBox $\mathcal{T}$ iff $\mathcal{T}$ is preserved under inverse p-morphisms iff such a rewriting exists at all.

Theorem 9 *An $\mathcal{ALCQI}$ TBox is* s*-conservatively $\mathcal{ALC}$-rewritable iff it is preserved under inverse p-morphisms.*

For m-conservative rewritability, we have:

Theorem 10 *If an $\mathcal{ALCQI}$ TBox is preserved under global i-bisimulations and inverse p-morphisms, then it is* m*-conservatively $\mathcal{ALC}$-rewritable.*

Proof. From preservation under global i-bisimulations of $\mathcal{T}$ follows the existence of an equivalent $\mathcal{ALCI}$ TBox $\mathcal{T}'$. Then $\mathcal{T}$ is m-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}'$ is m-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}'$ is preserved under inverse p-morphisms (Theorem 3). ❏

We conjecture that the converse also holds. By Lemma 1, m-conservatively $\mathcal{ALC}$-rewritable $\mathcal{ALCQI}$ TBoxes are preserved under inverse p-morphims. Thus, the conjecture would follow from preservation under global i-bisimulations.

6 Discussion and Future Work

Up to now, our focus has been on rewritability between expressive DLs. However, rewritability to the lightweight DLs from the *DL-Lite* and $\mathcal{EL}$ families is of great interest as well. Table 1 gives our model-theoretic characterization of rewritability from $\mathcal{ALCI}$ to *DL-Lite*$_{horn}$ (without role inclusions) [Calvanese et al., 2007; Artale et al., 2009]. The characterization of equivalent rewritability in terms of products and succ-simulations was given by Lutz, Piro, and Wolter [2011]. It is straightforward to prove that it also applies to m- and s-conservative rewritability of $\mathcal{ALCI}$ TBoxes by first showing that Theorem 1 holds for rewritings into *DL-Lite*$_{horn}$ as well:

Theorem 11 *For $\mathcal{ALCI}$ TBoxes, equivalent DL-Lite$_{horn}$-rewritability,* m*-conservative DL-Lite$_{horn}$-rewritability, as well as* s*-conservative DL-Lite$_{horn}$-rewritability coincide and are* EXPTIME*-complete.*

Rewritability into *DL-Lite* dialects with role inclusions (where Theorem 1 does not hold) and into $\mathcal{EL}$ appear to be much more challenging and a detailed study remains for future work. More generally, at the moment we only fully understand conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$-rewritability; in all other cases, it remains to determine the optimal size of rewritings, the complexity of computing them, as well as tight bounds for the complexity of deciding rewritability. Based on the resulting algorithms, it would be of great interest to study conservative rewritability in practice and, in particular, determine the rewritability status of real-world ontologies.

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A Proofs for Section 2

We prove an extension of Theorem 1 that covers *DL-Lite$_{horn}$* as well.

Theorem 12 *Let $\mathcal{L}$ be a DL reflecting disjoint unions, $\mathcal{T}$ an $\mathcal{L}$ TBox, and let $\mathcal{L}' \in \{\mathcal{ALCQ}, \mathcal{ALCI}, \mathcal{ALC}, \textit{DL-Lite}_{horn}\}$. If $\mathcal{T}$ is* m*-conservatively (or* s*-conservatively) $\mathcal{L}'$-rewritable, then $\mathcal{T}$ has a* m*-conservative (or, respectively,* s*-conservative) $\mathcal{L}'$-rewriting not using role names outside* $\mathsf{sig}(\mathcal{T})$.

Proof. The proof for $\mathcal{ALC}$ is given in the paper and the proofs for $\mathcal{ALCQ}$ and $\mathcal{ALCI}$ are straightforward extensions of the proof for $\mathcal{ALC}$. Thus, we give the proof for *DL-Lite$_{horn}$*. Let $\mathcal{T}'$ be a *DL-Lite$_{horn}$*-rewriting (it does not matter whether model or s-conservative) of $\mathcal{T}$ using fresh role names. Define $\mathcal{T}''$ as follows: introduce for every role name $r \in \mathsf{sig}(\mathcal{T}') \setminus \mathsf{sig}(\mathcal{T})$ fresh concept names $A_{\exists r.\top}$ and $A_{\exists r^-.\top}$. Denote by D^* the result of replacing any basic concept $\exists s.\top$ in D by $A_{\exists s.\top}$. Define $\mathcal{T}''$ by taking

- $C^* \sqsubseteq D^*$ for all $C \sqsubseteq D \in \mathcal{T}'$;
- $A_{\exists s.\top} \sqsubseteq \bot$ whenever $\exists s.\top$ occurs in $\mathcal{T}'$, s does not occur in $\mathcal{T}$, and $\exists s.\top$ is not satisfiable in a model of $\mathcal{T}'$.

We show that $\mathcal{T}''$ is an m-conservative (respectively, s-conservative) *DL-Lite$_{horn}$*-rewriting of $\mathcal{T}$ without additional role names.

It is sufficient to show that $\mathcal{T}'' \models \mathcal{T}$. Assume this is not the case. Let $\mathcal{I}$ be a model of $\mathcal{T}''$ which is not a model of $\mathcal{T}$. We may assume that $r^{\mathcal{I}} = \emptyset$ for all $r \notin \mathsf{sig}_R(\mathcal{T})$. Now define $\mathcal{J}$ as follows: for every $d \in \Delta^{\mathcal{I}}$ and $\exists r.\top \in \mathsf{sub}(\mathcal{T}')$ such that $d \in A^{\mathcal{I}}_{\exists r.\top}$ take a *universal* model $\mathcal{I}_{d,\exists r.\top}$ of $\mathcal{T}'$ and $\exists r^-.\top$ satisfying $\exists r^-.\top$ in its root $e_{d,\exists r.\top}$. Now define $\mathcal{J}$ by taking $\mathcal{I}$ and connecting d to $e_{d,\exists r.\top}$ using r. One can show the following:

For all $d \in \Delta^{\mathcal{I}}$ and all $C \in \mathsf{sub}(\mathcal{T}')$: $d \in (C^*)^{\mathcal{I}}$ iff $d \in C^{\mathcal{J}}$.

It follows that $\mathcal{J}$ is a model of $\mathcal{T}'$. Clearly, $\mathcal{J}$ refutes $\mathcal{T}$ since $\mathcal{I}$ refutes $\mathcal{T}$. We have derived a contradiction to the assumption that $\mathcal{T}' \models \mathcal{T}$. ❑

B $\mathcal{ALCI}$ to $\mathcal{ALC}$ Rewritability

We prove Theorem 2 in a number of steps.

Theorem 13 *An $\mathcal{ALCI}$ TBox $\mathcal{T}$ is* m*-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved generated subinterpretations. Moreover, if $\mathcal{T}$ is* m*-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}^\dagger$ is a model-conservative $\mathcal{ALC}$-rewriting.*

Proof. We show the following:

1. If an $\mathcal{ALCI}$ TBox $\mathcal{T}$ is m-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}$ is preserved under generated subinterpretations;
2. If an $\mathcal{ALCI}$ TBox $\mathcal{T}$ is preserved under generated subinterpretations, then $\mathcal{T}^\dagger$ is a m-conservative rewriting of $\mathcal{T}$.

1. Assume that $\mathcal{T}'$ is a m-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$. By Theorem 1, we can assume that all role names in $\mathcal{T}'$ also occur in $\mathcal{T}$. Assume for a proof by contradiction that $\mathcal{T}$ is not preserved under generated subinterpretations. Then there is a model $\mathcal{J}$ of $\mathcal{T}$ and a generated subinterpretation $\mathcal{I}$ of $\mathcal{J}$ that is not a model of $\mathcal{T}$. We also have a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}'$. Let $\mathcal{I}'$ be the restriction of $\mathcal{J}'$ to $\Delta^{\mathcal{I}}$. Since all role names in $\mathcal{T}'$ also occur in $\mathcal{T}$, we may assume that $r^{\mathcal{J}'} = \emptyset$ for all roles r that are not in $\mathcal{T}$. Consequently, $\mathcal{I}'$ is a generated subinterpretation of $\mathcal{J}'$ and, therefore, a model of $\mathcal{T}'$. We have $\mathcal{I}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$, and so $\mathcal{I}$ is a model of $\mathcal{T}$, which is a contradiction.

2. Suppose $\mathcal{T}$ is preserved under generated subinterpretations. We show that $\mathcal{T}^\dagger$ is a m-conservative rewriting of $\mathcal{T}$ (Claim 2 below). We first show an auxiliary claim. An interpretation $\mathcal{I}$ is called *proper* if $\bar{r}^{\mathcal{I}} = \{(d,e) \mid (e,d) \in r^{\mathcal{I}}\}$, for all fresh role names $\bar{r}$, and $B^{\mathcal{I}}_{\exists r.C} = (\exists r.C)^{\mathcal{I}}$, for all fresh concept names $B_{\exists r.C}$.

Claim 1. A proper interpretation is a model of $\mathcal{T}$ iff it is a model of $\mathcal{T}^\dagger$.

Proof sketch. Let $\mathcal{I}$ be proper. It is not hard to show that $C^{\mathcal{I}} = (C^\sharp)^{\mathcal{I}}$ for all $C \in \mathsf{sub}(\mathcal{T})$. This makes both the 'if' and the 'only if' directions easy to verify.

Claim 2. An interpretation $\mathcal{I}$ is a model of $\mathcal{T}$ iff there exists a model $\mathcal{I}'$ of $\mathcal{T}^\dagger$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

Proof. ($\Rightarrow$) Let $\mathcal{I}$ be a model of $\mathcal{T}$. Extend $\mathcal{I}$ to an interpretation $\mathcal{I}'$ by setting $B^{\mathcal{I}'}_{\exists r.C} = (\exists r.C)^{\mathcal{I}}$ for every fresh concept name $B_{\exists r.C}$ and $\bar{r}^{\mathcal{I}'} = (r^-)^{\mathcal{I}}$ for every fresh role name $\bar{r}$. Then $\mathcal{I}'$ is proper and, by Claim 1, a model of $\mathcal{T}^\dagger$. Moreover, we clearly have $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

($\Leftarrow$) Let $\mathcal{I}'$ be a model of $\mathcal{T}^\dagger$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$. Extend $\mathcal{I}'$ by setting $\bar{r}^{\mathcal{I}'} = \bar{r}^{\mathcal{I}'} \cup \{(d,e) \mid (e,d) \in r^{\mathcal{I}'}\}$ for every fresh role name $\bar{r}$, and denote the extended interpretation by $\mathcal{I}''$. It can be verified that $\mathcal{I}''$ is still a model of $\mathcal{T}^\dagger$. As an example, consider the CI $C^\sharp \sqsubseteq \forall \bar{r}.B_{\exists r.C}$. Assume that $(d,e) \in \bar{r}^{\mathcal{I}''} \setminus \bar{r}^{\mathcal{I}'}$ and $d \in (C^\sharp)^{\mathcal{I}''}$. Then $d \in (C^\sharp)^{\mathcal{I}'}$. It suffices to show that $e \in B^{\mathcal{I}'}_{\exists r.C}$, which follows from the facts that $(e,d) \in r^{\mathcal{I}'}$ and $\mathcal{I}' \models \exists r.C^\sharp \sqsubseteq B_{\exists r.C}$. We also note that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}''$.

We now further modify $\mathcal{I}''$ to an interpretation $\mathcal{J}$. Let $\mathcal{I}_0$ be the disjoint copy of $\mathcal{I}''$ in which every $d \in \Delta^{\mathcal{I}''}$ is renamed to d'. Then $\mathcal{J}$ is constructed by starting with the disjoint union of $\mathcal{I}''$ and $\mathcal{I}_0$ and then

1. replacing each edge $(d,e) \in \bar{r}^{\mathcal{J}}$ such that $d,e \in \Delta^{\mathcal{I}''}$ and $(e,d) \notin r^{\mathcal{J}}$ with the two edges $(e',d) \in r^{\mathcal{J}}$ and $(d,e') \in \bar{r}^{\mathcal{J}}$;
2. for each edge $(d',e') \in \bar{r}^{\mathcal{J}}$ such that $d',e' \in \Delta^{\mathcal{I}_0}$ and $(e',d') \notin r^{\mathcal{J}}$, adding the edge $(e',d') \in r^{\mathcal{J}}$.

It can be verified that $\mathcal{J}$ is still a model of $\mathcal{T}^\dagger$. Consequently, $\mathcal{J}$ is proper and, by Claim 1, a model of $\mathcal{T}$. Now let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by setting $s^{\mathcal{J}'}$ for all role names s that do not occur in $\mathcal{T}$ (including the role names $\bar{r}$). Clearly, $\mathcal{J}'$ is also a model of $\mathcal{T}$. Moreover, $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$ and the construction of $\mathcal{J}$ imply that $\mathcal{I}$ is a generated submodel of $\mathcal{J}'$. Since $\mathcal{T}$ is preserved under generated subinterpretations, we have $\mathcal{I} \models \mathcal{T}$ as required. ❑

We now show that s-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability coincides with m-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritablity. We employ robustness under replacement of $\mathcal{ALCI}$, which can be formulated as follows [Konev et al., 2009, Theorem 4] (we formulate the result for the pair $\mathcal{ALCQI}$ and $\mathcal{ALCQ}$ as well as we need it later):

Theorem 14 *Let $\mathcal{T}'$ be a s-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCI}$ TBox $\mathcal{T}$ not using role names not in $\mathsf{sig}(\mathcal{T})$. Let $\mathcal{T}''$ and $C \sqsubseteq D$ be in $\mathcal{ALCI}$ with $\mathsf{sig}(\mathcal{T}'', C \sqsubseteq D) \cap (\mathsf{sig}(\mathcal{T}') \setminus \mathsf{sig}(\mathcal{T})) = \emptyset$. Then $\mathcal{T}' \cup \mathcal{T}'' \models C \sqsubseteq D$ iff $\mathcal{T} \cup \mathcal{T}'' \models C \sqsubseteq D$.*

The same result holds if $\mathcal{ALCI}$ and $\mathcal{ALC}$ are replaced by $\mathcal{ALCQI}$ and $\mathcal{ALCQ}$, respectively.

In what follows we denote by $\mathsf{sig}_R(\mathcal{T})$ the set of role names in $\mathcal{T}$.

Theorem 15 *An $\mathcal{ALCI}$-TBox $\mathcal{T}$ is s-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is m-conservatively $\mathcal{ALC}$-rewritable.*

Proof. For a concept name A, we define inductively a *relativization* $C_{|A}$ of an $\mathcal{ALCI}$ concept C to A by taking:

$$\begin{aligned} B_{|A} &= B \sqcap A, \\ (\neg C)_{|A} &= A \sqcap \neg C_{|A}, \\ (C \sqcap D)_{|A} &= C_{|A} \sqcap D_{|A}, \\ (\exists r.C)_{|A} &= A \sqcap \exists r.(A \sqcap C_{|A}). \end{aligned}$$

For an interpretation $\mathcal{I}$ with $A^{\mathcal{I}} \neq \emptyset$, we denote by $\mathcal{I}_{|A}$ the subinterpretation of $\mathcal{I}$ with domain $A^{\mathcal{I}}$. We employ the following easily proved

Claim. For any interpretation $\mathcal{I}$, any $\mathcal{ALCI}$ concept C and any concept name A not in C, the following holds:

- $\mathcal{I}_{|A}$ is a generated subinterpretation of $\mathcal{I}$ iff $\mathcal{I} \models A \sqsubseteq \forall r.A$ for all $r \in \mathsf{N_R}$;
- for all $d \in \Delta^{\mathcal{I}}$, we have $d \in (C_{|A})^{\mathcal{I}}$ iff $d \in C^{\mathcal{I}_{|A}}$.

Now suppose $\mathcal{T}$ has a s-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$, but is not preserved under generated subinterpretations. By Theorem 1, we may assume that $\mathcal{T}'$ uses no additional role names. Then we have for $A \in \mathsf{N_C} \setminus \mathsf{sig}(\mathcal{T})$:

$$\mathcal{T} \cup \{A \sqsubseteq \forall r.A \mid r \in \mathsf{sig}_R(\mathcal{T})\} \not\models C|_A \sqsubseteq D|_A,$$

for some $(C \sqsubseteq D) \in \mathcal{T}$. Thus, by Theorem 14,

$$\mathcal{T}' \cup \{A \sqsubseteq \forall r.A \mid r \in \mathsf{sig}_R(\mathcal{T})\} \not\models C|_A \sqsubseteq D|_A.$$

Take a model $\mathcal{I}$ of $\mathcal{T}' \cup \{A \sqsubseteq \forall r.A \mid r \in \mathsf{sig}_R(\mathcal{T})\}$ such that $\mathcal{I} \not\models C|_A \sqsubseteq D|_A$. Then $\mathcal{I}_{|A}$ is a model of $\mathcal{T}'$ such that $C \sqsubseteq D$, which is impossible. ❑

C ω-saturated interpretations and unfoldings

We analyze the relevant properties of ω-saturated interpretations and unfoldings of interpretations. Given a DL $\mathcal{L}$ and a signature Σ, a $\mathcal{L}_\Sigma$-type $\mathbf{t}$ is a maximal satisfiable subset of the set of $\mathcal{L}_\Sigma$-concepts. For an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$ we set

$$t_{\mathcal{I}}^{\mathcal{L}_\Sigma}(d) = \{C \in \mathcal{L}_\Sigma \mid d \in C^{\mathcal{I}}\}$$

Let r be a role and let $\mathbf{t}, \mathbf{t}'$ be $\mathcal{L}_\Sigma$-types. We write $\mathbf{t} \leadsto_r^{\mathcal{L}_\Sigma} \mathbf{t}'$ if

- $\forall r.C \in \mathbf{t}$ implies $C \in \mathbf{t}'$ for all $\forall r.C \in \mathcal{L}_\Sigma$;
- $\forall r^-.C \in \mathbf{t}'$ implies $C \in \mathbf{t}$ for all $\forall r^-.C \in \mathcal{L}_\Sigma$.

An interpretation $\mathcal{I}$ is *counting $\mathcal{L}_\Sigma$-saturated for a role r* if for all $d \in \Delta^{\mathcal{I}}$ and all $\mathcal{L}_\Sigma$-types $\mathbf{t}$: if $(\geq n\ r\ C) \in t_{\mathcal{I}}^{\mathcal{L}_\Sigma}(d)$ for all $C \in \mathbf{t}$ then there exist at least n many $d' \in \Delta^{\mathcal{I}}$ with $(d, d') \in r^{\mathcal{I}}$ and $t_{\mathcal{I}}^{\mathcal{L}_\Sigma}(d') = \mathbf{t}$ (in DLs without qualified number restrictions we use $(\geq 1\ r\ C)$ as an abbreviation for $\exists r.C$ here).

Lemma 3 *Every ω-saturated interpretation is counting $\mathcal{L}_\Sigma$-saturated for every role r.*

Lemma 3 has a number of consequences we use throughout this paper:

- if $\mathcal{L}$ is $\mathcal{ALCQ}$ and r is a role name, then any two nodes in an interpretation $\mathcal{I}$ that satisfy the same $\mathcal{L}_\Sigma$-type, have for any given $\mathcal{L}_\Sigma$-type $\mathbf{t}$ either both infinitely many r-successors satisfying $\mathbf{t}$ or have exactly the same number of r-successors satisfying $\mathbf{t}$.
- if $\mathcal{L}$ is $\mathcal{ALCQI}$, then this holds for inverse roles as well.

We also use the fact that standard unfoldings of interpretations into tree interpretations preserve counting $\mathcal{L}_\Sigma$-saturatedness. Here, given an interpretation $\mathcal{I}$, the tree-unfolding $\mathcal{I}^*$ of $\mathcal{I}$ at d_0 has as its domain the set $\Delta^{\mathcal{I}^*}$ of all words $d_0 r_1 d_1 \cdots r_n d_n$ such that the r_i are roles and

- $(d_i, d_{i+1}) \in r_{i+1}^{\mathcal{I}}$ for all $i < n$;
- $(d_{i-1}, r_i^-) \neq (d_{i+1}, r_{i+1})$ for all $i < n$.

and $(\mathsf{tail}(w)$ denotes the last element of word $w)$:

- for every concept name A, $w \in A^{\mathcal{I}^*}$ iff $\mathsf{tail}(w) \in A^{\mathcal{I}}$;
- for every role name s, $(w, wrd) \in s^{\mathcal{I}^*}$ iff $r = s$ is a role name and $(\mathsf{tail}(w), d) \in r^{\mathcal{I}}$ or $r = s^-$ is an inverse role and $(d, \mathsf{tail}(w)) \in r^{\mathcal{I}}$.

D $\mathcal{ALCQI}$ to $\mathcal{ALCQ}$-Rewritability

We present the proof of Theorem 4.

We first define the rewriting $\mathcal{T}'$ in more detail. Assume an $\mathcal{ALCQI}$ TBox $\mathcal{T}$ is given. We assume all qualified number restrictions are of the form $(\leq n\ r\ C)$, with r a role. We use $\exists r.C$, $\forall r.C$, $(\geq n\ r\ C)$ as abbreviations. Introduce for each $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$ concept $D = (\leq n\ r^-\ D')$ with r a role name a fresh concept name B_D and denote by Σ the set of all concept and role names in $\mathsf{sig}(\mathcal{T})$ and the additional B_D. For any concept C in $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$ we denote by $C^\uparrow$ the result of replacing any top-most occurrence of $D = (\leq n\ r^-\ D')$ in C by B_D. Conversely, by $C^\downarrow$ we denote the result when all B_D are replaced by D. For any type $\mathbf{t}$ we set

$$\mathbf{t}^\uparrow = \{D^\uparrow \mid D \in \mathbf{t}\}$$

and similarly for $\mathbf{t}^\downarrow$.

Now define $\mathcal{T}'$ as the set of CIs $C^\uparrow \sqsubseteq D^\uparrow$ such that $\mathcal{T} \models C \sqsubseteq D$.

We show that $\mathcal{T}'$ is a model-conservative rewriting of $\mathcal{T}$ if $\mathcal{T}$ is preserved under inverse p-morphisms. Clearly for every model $\mathcal{I}$ of $\mathcal{T}$ there exists a model $\mathcal{J}$ of $\mathcal{T}'$ such that $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. For the proof of $\mathcal{T}' \models \mathcal{T}$ the following relationship between $\mathcal{T}$ and $\mathcal{T}'$ is fundamental.

Lemma 4 *Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-types. Then the following hold:*

- *$\mathbf{t}_1$ is satisfiable relative to $\mathcal{T}$ iff $\mathbf{t}_1^{\uparrow}$ is satisfiable relative to $\mathcal{T}'$;*
- *for all roles r:*

$$\mathbf{t}_1 \rightsquigarrow_r^{\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}} \mathbf{t}_2 \quad \Leftrightarrow \quad \mathbf{t}_1^{\uparrow} \rightsquigarrow_r^{\mathcal{ALCQ}_\Sigma} \mathbf{t}_2^{\uparrow}$$

Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be $\mathcal{ALCQ}_\Sigma$-types. Then the following hold:

- *$\mathbf{t}_1$ is satisfiable relative to $\mathcal{T}'$ iff $\mathbf{t}_1^{\downarrow}$ is satisfiable relative to $\mathcal{T}$;*
- *for all roles r:*

$$\mathbf{t}_1 \rightsquigarrow_r^{\mathcal{ALCQ}_\Sigma} \mathbf{t}_2 \quad \Leftrightarrow \quad \mathbf{t}_1^{\downarrow} \rightsquigarrow_r^{\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}} \mathbf{t}_2^{\downarrow}$$

We now prove a series of lemmata that allow us to prove that $\mathcal{T}' \models \mathcal{T}$.

Lemma 5 *Assume $\mathcal{I}$ is a model of $\mathcal{T}'$ such that $(*)$ for every $d \in \Delta^{\mathcal{I}}$ and $\mathcal{ALCQ}_\Sigma$-type $\mathbf{t}$ that is satisfiable relative $\mathcal{T}'$ with*

$$t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d) \rightsquigarrow_{r^-}^{\mathcal{ALCQ}_\Sigma} \mathbf{t}$$

there exists exactly one d' with $(d', d) \in r^{\mathcal{I}}$ such that $t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d') = \mathbf{t}$. Then there exists a model $\mathcal{J}$ of $\mathcal{T}$ such that $\mathcal{I}$ is a generated subinterpretation of $\mathcal{J}$.

Proof. For each $d \in \Delta^{\mathcal{I}}$ consider the type $\mathbf{t}_d = t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d)$ and take an unfolding of an ω-saturated model $\mathcal{I}_d$ of $\mathcal{T}$ satisfying $\mathbf{t}_d^{\downarrow}$ in a node $d_{\mathbf{t}}$. Remove from $\mathcal{I}_d$

- all $r^{\mathcal{I}_d}$-successors (together with their subtrees) of $d_{\mathbf{t}}$, for any role name r;
- for each role name r and each $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-type $\mathbf{t}$ with

$$\mathbf{t}_d^{\downarrow} \rightsquigarrow_{r^-}^{\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}} \mathbf{t}$$

  exactly one $r^{\mathcal{I}_d}$-predecessor (together with its subtree) of $d_{\mathbf{t}}$ satisfying $\mathbf{t}$.

Denote the resulting interpretation by $\mathcal{I}_d'$ and hook it to $\mathcal{I}$ by identifying $d_{\mathbf{t}}$ and d. The resulting interpretation, $\mathcal{J}$, is a model of $\mathcal{T}$. To see this it is sufficient to show the following:

Claim 1. For every $D = (\leq n\ r\ C)$ and $d \in \Delta^{\mathcal{I}}$: $d \in D^{\mathcal{J}}$ iff $d \in B_D^{\mathcal{I}}$.

The proof of Claim 1 is straightforward and uses Lemma 3 and Lemma 4. ❑

The following result shows that the assumption $\mathcal{T}' \not\models \mathcal{T}$ leads to a contradiction if $\mathcal{T}$ is preserved under inverse counting p-morphisms.

Lemma 6 *Assume $\mathcal{T}' \not\models \mathcal{T}$ and $\mathcal{T}$ is preserved under inverse counting p-morphisms. Then there exists a model of $\mathcal{T}'$ satisfying $(*)$ that is not a model of $\mathcal{T}$.*

Proof. Assume $\mathcal{T}' \not\models \mathcal{T}$. By taking an ω-saturated witness and unfolding it we obtain a model $\mathcal{I}$ of $\mathcal{T}'$ that is not a model of $\mathcal{T}$ such that

- $\mathcal{I}$ is counting $\mathcal{ALCQ}_\Sigma$-saturated for every role r;
- $\mathcal{I}$ is counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r.

We can next ensure that $\mathcal{I}$ satisfies the following version $(*^\infty)$ of $(*)$:

$(*^\infty)$ for every $d \in \Delta^{\mathcal{I}}$ and $\mathcal{ALCQ}_\Sigma$-type $\mathbf{t}$ that is satisfiable relative $\mathcal{T}'$ with

$$t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d) \rightsquigarrow_{r^-}^{\mathcal{ALCQ}_\Sigma} \mathbf{t}$$

there exist infinitely many d' with $(d', d) \in r^{\mathcal{I}}$ such that $t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d') = \mathbf{t}$.

For each $d \in \Delta^{\mathcal{I}}$ consider the type $\mathbf{t}_d = t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d)$ and take an unfolding of an ω-saturated model $\mathcal{I}_d$ of $\mathcal{T}$ satisfying $\mathbf{t}_d^{\downarrow}$ in a node $d_{\mathbf{t}}$. Remove from $\mathcal{I}_d$ all $r^{\mathcal{I}_d}$-successors (together with their subtrees) of $d_{\mathbf{t}}$, for any role name r. Extend the resulting interpretation by setting $B_D^{\mathcal{I}_d} = D^{\mathcal{I}_d}$ and hook it to $\mathcal{I}$ by identifying $d_{\mathbf{t}}$ and d. The resulting interpretation $\mathcal{I}'$ is a model of $\mathcal{T}'$ but still not a model of $\mathcal{T}$ since $\mathcal{I}$ is a generated subinterpretation of $\mathcal{I}'$. We can do the same construction ω times and obtain an interpretation with $(*^\infty)$.

It is now straightforward to manipulate the interpretation in such a way that in addition the following holds:

$(=^{\mathbf{t}})$ any two nodes satisfying the same $\mathcal{ALCQ}_\Sigma$-types have exactly the same number of r-successors satisfying a given $\mathcal{ALCQ}_\Sigma$-type.

We are now in the position to define an interpretation $\mathcal{I}_p$ satisfying $(*)$ such that there is a counting p-morphism from $\mathcal{I}$ to $\mathcal{I}_p$. We may assume that for

$$<= \bigcup_{r \text{ a role}} r^{\mathcal{I}}$$

the graph $(\Delta^{\mathcal{I}}, <)$ is a tree (of possibly uncountable outdegree) with root d_0 and $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$ for any two distinct roles r, s. Denote by $\Delta_{=n}^{\mathcal{I}}$ the nodes of distance n from d_0 and by $\Delta_{\leq n}^{\mathcal{I}}$ the nodes of distance at most n from d_0.

We define $\mathcal{I}_p$ inductively. Let $\Gamma_0 = \{d_0\}$, and assume $\Gamma_n \subseteq \Delta_{\leq n}^{\mathcal{I}}$ has been defined. We include Γ_n in Γ_{n+1} and, in addition, for each $d \in \Gamma_n \cap \Delta_{=n}^{\mathcal{I}}$ and role name r we include in Γ_{n+1}

- all d' with $d < d'$ and $(d, d') \in r^{\mathcal{I}}$;
- for each $\mathcal{ALCQ}_\Sigma$-type $\mathbf{t}$ with

$$t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d) \rightsquigarrow_{r^-}^{\mathcal{ALCQ}_\Sigma} \mathbf{t}$$

  exactly one d' with $(d', d) \in r^{\mathcal{I}}$ and $t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d') = \mathbf{t}$. If there exists such a d' with $d' < d$, then d' is contained in Γ_n already and no additional d' is added to Γ_{n+1}.

Now let $\mathcal{I}_p$ be the subinterpretation of $\mathcal{I}$ induced by $\Gamma = \bigcup_{n<\omega} \Gamma_n$. Clearly it still satisfies $\mathcal{T}'$. To show that it does not satisfy $\mathcal{T}$, we construct a bounded counting p-morphism f from $\mathcal{I}$ onto $\mathcal{I}_p$. We construct f as the union $\bigcup_{n\geq 0} f_n$ of mappings f_n with domain $\Delta_{\leq n}^{\mathcal{I}}$ as follows: We set $f_0(d_0) = d_0$. Assume f_n has been defined on $\Delta_{\leq n}^{\mathcal{I}}$ and assume f_n has the following properties for all $d, d' \in \Delta_{\leq n}^{\mathcal{I}}$:

- $t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d) = t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(f_n(d))$;
- if $(d, d') \in r^{\mathcal{I}}$, then $(f_n(d), f_n(d')) \in r^{\mathcal{I}_p}$.

For each $d \in \Delta^{\mathcal{I}}_{=n}$ and *role name* r let $f_{r,d}$ be a bijection from the $r^{\mathcal{I}}$-successors of d onto the $r^{\mathcal{I}_p}$-successors of $f_n(d)$ such that $t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(e) = t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(f_{r,d}(e))$ for all e. Note that all $r^{\mathcal{I}}$-successors of $f_n(d)$ are in $\Delta^{\mathcal{I}_p}$ and that if $f_n(d')$ has been defined already for an $r^{\mathcal{I}}$-successor d' of d, then $d' < d$ (and so there is at most one such d') and we set $f_{r,d}(d') = f_n(d)$.

For each r^--successor d' of d there exists by construction exactly one r^--successor e of $f_n(d)$ in $\Delta^{\mathcal{I}_p}$ such that $t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(d') = t_{\mathcal{I}}^{\mathcal{ALCQ}_\Sigma}(e)$. Let $f_{r^-,d}(d') = e$. We set

$$f_{n+1} = f_n \cup \bigcup_{d \in \Delta^{\mathcal{I}_1}_{\leq n}} f_{r,d} \cup f_{r^-,d}.$$

Finally, let $f = \bigcup_{n \geq 0} f_n$. ❑

We have thus established the following result.

Theorem 16 *If $\mathcal{T}$ is preserved under inverse counting p-morphisms, then $\mathcal{T}'$ is a* m*-conservative rewriting of $\mathcal{T}$.*

Our aim now is to establish the following result.

Theorem 17 *Let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox. If $\mathcal{T}$ is* s*-conservatively $\mathcal{ALCQ}$-rewritable, then $\mathcal{T}$ is* m*-conservatively $\mathcal{ALCQ}$-rewritable.*

Let $\mathcal{T}^*$ be a s-conservative $\mathcal{ALCQ}$-rewriting of $\mathcal{T}$. We may assume that $\mathcal{T}^*$ uses no fresh role names. To prove Theorem 17 it is sufficient to prove that $\mathcal{T}' \models \mathcal{T}$. We show this by proving the following:

- $\mathcal{T}$ is preserved under generated subinterpretations;
- Lemma 5 still holds;
- Assume $\mathcal{T}' \not\models \mathcal{T}$. Then there exists a model of $\mathcal{T}'$ satisfying $(*)$ that is not a model of $\mathcal{T}$.

Point 1 can be proved using Theorem 14. Point 2 holds because in the proof of Lemma 5 only preservation under generated subinterpretations of $\mathcal{T}$ was used. For Point 3 we consider the proof of Lemma 6. We have a counting p-morphism f from the model $\mathcal{I}$ of $\mathcal{T}'$ to $\mathcal{I}_p$, where $\mathcal{I} \not\models \mathcal{T}$. Thus, if we can show

(†) there exists a model $\mathcal{I}_p^*$ of $\mathcal{T}^*$ such that $\mathcal{I}_p^* =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_p$

we are done: using f we obtain that there exists a model $\mathcal{I}^*$ of $\mathcal{T}^*$ such that $\mathcal{I}^* =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. But then, since $\mathcal{T}^* \models \mathcal{T}$, we have that $\mathcal{I}$ is a model of $\mathcal{T}$, a contradiction. To prove (†), observe that the proof of Lemma 5 show that there exists a model $\mathcal{J}$ of $\mathcal{T}$ that is counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r such that $\mathcal{I}_p$ is a generated subinterpretation of $\mathcal{J}$. Since $\mathcal{T}^*$ is preserved under generated subinterpretations it is therefore sufficient to prove:

Lemma 7 *Let $\mathcal{J}$ be a tree-model of $\mathcal{T}$ that is counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r. Then there exists a model $\mathcal{J}^*$ to $\mathcal{T}^*$ such that $\mathcal{J}^* =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$.*

Proof. Let d be a root of $\mathcal{J}$. The type $t_{\mathcal{J}}^{\mathcal{ALCQI}_\Sigma}(d)$ is satisfiable relative to $\mathcal{T}^*$. Thus we find an ω-saturated model $\mathcal{J}'$ of $\mathcal{T}^*$ such that $\mathcal{J}$ is a subinterpretation of $\mathcal{J}'$ and for each $d' \in \Delta^{\mathcal{I}}$, $t_{\mathcal{J}}^{\mathcal{ALCQI}_\Sigma}(d) = t_{\mathcal{J}'}^{\mathcal{ALCQI}_\Sigma}(d)$. We can assume (by unfolding) that $\mathcal{J}'$ is a tree-interpretation as well. Now one can use selective filtration over the subconcepts of $\mathcal{T}^*$ to construct a $\mathcal{J}^*$ with the required properties. ❑

E $\mathcal{ALCQ}$ to $\mathcal{ALC}$-rewritability

Lemma 2 An $\mathcal{ALC}$ TBox $\mathcal{T}'$ is a s-conservative rewriting of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ iff $\mathcal{T}'$ is a m-conservative rewriting of $\mathcal{T}$ over the class of ditree interpretations of finite outdegree.

Proof. ($\Leftarrow$) We have to show that $\mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T}' \models C \sqsubseteq D$ for all $\mathcal{ALCQ}$-concepts C, D in the signature $\mathsf{sig}(\mathcal{T})$. In the 'if' direction, $\mathcal{T} \not\models C \sqsubseteq D$ implies that there is a model $\mathcal{I}$ of $\mathcal{T}$ with $\mathcal{I} \not\models C \sqsubseteq D$. We can always assume that $\mathcal{I}$ is a ditree interpretation of finite outdegree. Consequently, there is a directed tree model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. Thus, $\mathcal{J} \not\models C \sqsubseteq D$, and so $\mathcal{T}' \not\models C \sqsubseteq D$. The converse direction is similar.

($\Rightarrow$) Let $\mathcal{I}$ be a ditree model of $\mathcal{T}$ of finite outdegree with root d_0. We have to show that there is a model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. For every $d \in \Delta^{\mathcal{I}}$ and $i \geq 0$, set

$$\begin{aligned} C_d^0 &= \bigsqcap_{A \in \mathsf{sig}(\mathcal{T}), d \in A^{\mathcal{I}}} A \sqcap \bigsqcap_{A \in \mathsf{sig}(\mathcal{T}), d \notin A^{\mathcal{I}}} \neg A, \\ C_d^{i+1} &= C_d^i \sqcap \bigsqcap_{r \in \mathsf{sig}(\mathcal{T}), (d,e) \in r^{\mathcal{I}}} (= n_{d,r,C_e^i}\ r C_e^i) \sqcap \\ & \bigsqcap_{r \in \mathsf{sig}(\mathcal{T})} \forall r. \bigsqcup_{(d,e) \in r^{\mathcal{I}}} C_e^i, \end{aligned}$$

where $n_{d,r,C}$ is the cardinality of $\{(d,e) \in r^{\mathcal{I}} \mid e \in C^{\mathcal{I}}\}$. Let $\Gamma_{\mathcal{I}} = \{C_{d_0}^i \mid i \geq 0\}$. One can show that a tree interpretation $\mathcal{J}$ satisfies $\Gamma_{\mathcal{I}}$ at the root iff $\mathcal{J} \models_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. Since $\mathcal{T}'$ is a s-conservative rewriting of $\mathcal{T}$ and by compactness, Γ_d is satisfiable w.r.t. $\mathcal{T}'$. Clearly, any ditree model $\mathcal{J}$ of $\mathcal{T}'$ that satisfies Γ_d at the root is as required. The converse direction is similar. ❑

We split the proof of Theorem 5 into two parts.

Lemma 8 *An $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is* s*-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved under inverse p-morphisms over the class of ditree interpretations of finite outdegree. Moreover, if $\mathcal{T}$ is* s*-conservatively $\mathcal{ALC}$-rewritable (over this class), then $\mathcal{T}^\dagger$ is an (infinite) rewriting.*

Proof. ($\Rightarrow$) Suppose $\mathcal{T}$ has a s-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$, which only contains fresh concept names, but no fresh role names. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ditree interpretations of finite outdegree such that there is p-morphism f from $\mathcal{I}_1$ to $\mathcal{I}_2$ and $\mathcal{I}_2$ is a model of $\mathcal{T}$. We have to show that $\mathcal{I}_1$ is a model of $\mathcal{T}$. By Lemma 2, there is a model $\mathcal{J}_2$ of $\mathcal{T}'$ with $\mathcal{J}_2 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_2$. Clearly, f is also a p-morphism f from $\mathcal{I}_1$ to $\mathcal{J}_2$. By Lemma 1 and since $\mathcal{T}'$ contains only fresh concept names, we find a model $\mathcal{J}_1$ of $\mathcal{T}'$ such that $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1$. Consequently, $\mathcal{I}_1$ is a model of $\mathcal{T}$.

($\Leftarrow$) Assume that $\mathcal{T}$ is preserved under inverse p-morphisms on the class of ditree interpretations of finite outdegree. We show the following, which clearly implies that $\mathcal{T}^\dagger$ is an (infinite) $\mathcal{ALC}$-rewriting of $\mathcal{T}$.

1. if $\mathcal{T}^\dagger \models C \sqsubseteq D$ then $\mathcal{T} \models C \sqsubseteq D$ for all $\mathcal{ALCQ}$ inclusions $C \sqsubseteq D$ in $\mathsf{sig}(\mathcal{T})$;
2. $\mathcal{T}^\dagger \models \mathcal{T}$.

To obtain a finite $\mathcal{ALC}$-rewriting of $\mathcal{T}$, it then remains to invoke compactness: there is a finite subset $\mathcal{T}^\ddagger$ of $\mathcal{T}^\dagger$ such that $\mathcal{T}^\ddagger \models \mathcal{T}$. Clearly, $\mathcal{T}^\ddagger$ is as required.

Proof of Point 1. Assume $\mathcal{T} \not\models C \sqsubseteq D$ for some $\mathcal{ALCQ}$ inclusion $C \sqsubseteq D$ over $\mathsf{sig}(\mathcal{T})$. We find a ditree interpretation $\mathcal{I}$ that is a model of $\mathcal{T}$ such that $\mathcal{I} \not\models C \sqsubseteq D$. Define $\mathcal{I}'$ in the same way as $\mathcal{I}$ except that $B_D^{\mathcal{I}'} = D^{\mathcal{I}}$ for all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$ and that for $d \in D^{\mathcal{I}}$ we make $B_1^D, \dots, B_n^D$ true in distinct r-successor of d in which C holds. It is readily checked that $\mathcal{I}'$ is a model of $\mathcal{T}^\dagger$. Thus, $\mathcal{T}^\dagger \not\models C \sqsubseteq D$.

Proof of Point 2. Assume that $\mathcal{T}^\dagger \not\models \mathcal{T}$. Take a ditree interpretation $\mathcal{I}$ satisfying $\mathcal{T}^\dagger$ and refuting $\mathcal{T}$ in its root. First we manipulate $\mathcal{I}$ so that it has finite outdegree.

Clearly, we find a subinterpretation $\mathcal{I}'$ of $\mathcal{I}$ of finite outdegree that refutes $\mathcal{T}$. We have to be careful, however, to ensure that it still satisfies $\mathcal{T}^\dagger$. In particular, we have to ensure that no non-bisimilar successor nodes are introduced when removing nodes from $\mathcal{I}$.

We define $\mathcal{I}'$ as the limit of a sequence $\mathcal{I}_0, \mathcal{I}_1, \dots$ of interpretations:

- Set $\mathcal{I}_0 := \mathcal{I}$;
- Assume $\mathcal{I}_n$ has been defined. Let $\sim_n$ be the Σ-bisimulation relation on points of level n in $\mathcal{I}_n$. Let $[d]_{\sim_n}$ be the equivalence class of d w.r.t. $\sim_n$. Choose for every $D = (\geqslant m\ r\ C) \in \mathsf{sub}(\mathcal{T})$ and $e \in D^{\mathcal{I}_n}$ of level n in $\mathcal{I}_n$, m r-successors $d_1, \dots, d_m \in C^{\mathcal{I}_n}$ and include all $\mathcal{I}_{d_i}$ in $\mathcal{I}_{n+1}$. Also, choose for every $e \in B_D^{\mathcal{I}_n}$ of level n in $\mathcal{I}_n$ r-successors $d_i' \in (B_i^D \sqcap C)^{\mathcal{I}_n}$ for $1 \le i \le m$ and include all $\mathcal{I}_{d_i'}$ in $\mathcal{I}_{n+1}$.

  Finally, select for every $e \in [d]_{\sim_n}$ and every selected r-successor f of e for every $e' \in [d]_{\sim_n}$ an r-successor f' of e' that is Σ-bisimilar to f and include $\mathcal{I}_{f'}$ in $\mathcal{I}_{n+1}$ as well. This concludes the definition of $\mathcal{I}_{n+1}$.

Let $\mathcal{I}'$ be the intersection over all $\mathcal{I}_n$. $\mathcal{I}'$ has finite outdegree and clearly refutes $\mathcal{T}$. It remains to show that it is a model of $\mathcal{T}^\dagger$. The interesting inclusions are $\bigsqcap_{1\le i\le n}(\exists r.(C \sqcap C_i \sqcap \bigsqcap_{j\neq i} \neg C_j)) \sqsubseteq B_D$. To show that these are still true in $\mathcal{I}'$ it is sufficient to show that if d, d' are Σ-bisimilar r-successor of d in $\mathcal{I}$ and are included in $\Delta^{\mathcal{I}'}$, then they are Σ-bisimilar in $\mathcal{I}'$. But this is the case by construction.

We now define an interpretation $\mathcal{J}$ as the image of $\mathcal{I}'$ under a p-morphism. Let $[d]$ denote the set of all nodes of the same level as d that are Σ-bisimilar d. The domain of $\mathcal{J}$ consists of all words $[d_0]r_1[d_1]\cdots r_n[d_n]$, where d_0 is the root of $\mathcal{I}'$ and for all i there exist $e_i \in [d_i]$ and $e_{i+1} \in [d_{i+1}]$ such that $(e_i, e_{i+1}) \in r_{i+1}^{\mathcal{I}'}$. Set $[d_0]r_1[d_1]\cdots r_n[d_n] \in A^{\mathcal{J}}$ iff $d_n \in A^{\mathcal{I}'}$ and set $([d_0]r_1[d_1]\cdots r_n[d_n], [d_0]r_1[d_1]\cdots r_n[d_n]r_{n+1}[d_{n+1}]) \in r^{\mathcal{J}}$ iff $r = r_{n+1}$ and there exists $e_n \in [d_n]$ and $e_{n+1} \in [d_{n+1}]$ such that $(e_n, e_{n+1}) \in r^{\mathcal{I}'}$. This defines $\mathcal{J}$. Now one can show

- $f : d \mapsto [d]$ is a p-morphism from $\mathcal{I}'$ to $\mathcal{J}$;
- in $\mathcal{J}$, any Σ-bisimilar r-successors of a node are identical;
- $\mathcal{J} \models \mathcal{T}^\dagger$;
- $\mathcal{J} \models \mathcal{T}$.

The first three points are straightforward. Now for the final Point: we show by induction that $B_D^{\mathcal{J}} = D^{\mathcal{J}}$ for all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$. If $d \in B_D^{\mathcal{J}}$, then $d \in D^{\mathcal{J}}$ holds since $\mathcal{J}$ is a model of $\mathcal{T}^\dagger$ and the the first set of inclusions in $\mathcal{T}^\dagger$. Conversely, assume $d \in D^{\mathcal{J}}$. Then d has n distinct r-successors $d_1, \dots, d_n \in C^{\mathcal{J}}$. None of them is Σ-bisimilar, by Point 2. Thus, there are concepts $C_1, \dots, C_n$ in $\mathcal{ALC}$ and using symbols from Σ such that $d_i \in C_j^{\mathcal{J}'}$ iff $j = i$. By the final set of inclusions in $\mathcal{T}^\dagger$ we have $d \in B_D^{\mathcal{J}}$.

We obtain that $\mathcal{I}'$ is a model of $\mathcal{T}$ since $\mathcal{J}$ is a model of $\mathcal{T}$. But that is a contradiction. ❑

Theorem 5 The following conditions are equivalent for any $\mathcal{ALCQ}$ TBox $\mathcal{T}$:

1. $\mathcal{T}$ is s-conservatively $\mathcal{ALC}$-rewritable;
2. $\mathcal{T}$ is preserved under inverse p-morphisms;
3. $\mathcal{T}^\dagger$ is a s-conservative rewriting of $\mathcal{T}$.

Proof. It remains to prove (1) $\Rightarrow$ (2). Suppose an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is s-conservatively $\mathcal{ALC}$-rewritable. Consider interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_2 \models \mathcal{T}$ and there is a p-morphism f from $\mathcal{I}_1$ to $\mathcal{I}_2$. We have to show that $\mathcal{I}_1 \models \mathcal{T}$. Let $\mathcal{I}_1^\dagger$ and $\mathcal{I}_2^\dagger$ be the unfoldings of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Note that $\mathcal{I}_i \models \mathcal{T}$ iff $\mathcal{I}_i^\dagger \models \mathcal{T}$ and that we can lift f to a p-morphism $f^\dagger$ from $\mathcal{I}_1^\dagger$ to $\mathcal{I}_2^\dagger$. It is therefore sufficient to show $\mathcal{I}_1^\dagger \models \mathcal{T}$. By Theorem 8, $\mathcal{T}^\dagger \models \mathcal{T}$, and so $\mathcal{I}_1^\dagger \models \mathcal{T}$ if there exists a model $\mathcal{J}_1$ of $\mathcal{T}^\dagger$ such that $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1^\dagger$. By Lemma 1, such a model $\mathcal{J}_1$ exists if there exists a model $\mathcal{J}_2$ of $\mathcal{T}^\dagger$ with $\mathcal{J}_2 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_2^\dagger$. But the latter is straightforward using that $\mathcal{I}_2^\dagger$ is a model of $\mathcal{T}$. ❑

E.1 Proof of Theorem 7

We start with a sketch of the proof of Theorem 7 and then give a detailed proof. Assume, for a proof by contradiction, that $\mathcal{T}$ is an $\mathcal{ALCQ}$ TBox which is not preserved under global bisimulations but has a model-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$. By Theorem 1, we may also assume that $\mathcal{T}'$ contains no new role names. It follows that there is a model $\mathcal{I}$ of $\mathcal{T}$ and an interpretation $\mathcal{J}$, which is not a model of $\mathcal{T}$, such that $\mathcal{I}$ and $\mathcal{J}$ are globally bisimilar. Then one can show the following:

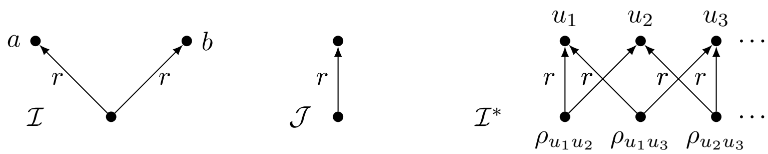
($*$) there exist ditree interpretations $\mathcal{I}$ and $\mathcal{J}$ that a globally bisimilar such that $\mathcal{I}$ is a model of $\mathcal{T}$, $\mathcal{J}$ is not a model of $\mathcal{T}$, and, for any $d \in \Delta^{\mathcal{I}}$, there are no distinct d_1, d_2 with $(d, d_1) \in r^{\mathcal{I}}$ and $(d, d_2) \in r^{\mathcal{I}}$ such that $(\mathcal{I}, d_1)$ and $(\mathcal{I}, d_2)$ are bisimilar.

Observe first that ($*$) leads to a contradiction. It is easy to see that any bisimulation S witnessing ($*$) gives rise to a p-morphism f from $\mathcal{J}$ to $\mathcal{I}$. Indeed, we may assume that S is a *level-bisimulation* in the sense that if $(d, e) \in S$, then the distance from d to the root of $\mathcal{I}$ is the same as the distance of e to the root of $\mathcal{J}$. The condition for $\mathcal{I}$ in ($*$) implies then that S is a function.

We now apply Lemma 1 to the p-morphism f from $\mathcal{J}$ to $\mathcal{I}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we find $\mathcal{I}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$ such that

$\mathcal{I}'$ is a model of $\mathcal{T}'$. Clearly, f is still a p-morphism from $\mathcal{J}$ to $\mathcal{I}'$. By Lemma 1, there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. But then $\mathcal{J}$ is a model of $\mathcal{T}$, and we have obtained a contradiction.

To illustrate the idea behind property $(*)$, consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ in the picture below, assuming that $\mathcal{I}$ is a model of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$.



The model $\mathcal{I}$ does not satisfy $(*)$ because $(\mathcal{I}, a)$ and $(\mathcal{I}, b)$ are bisimilar. To find a model of $\mathcal{T}$ for which $(*)$ holds, we construct the interpretation $\mathcal{I}^*$ shown on the right-hand side of the picture, where each pair of distinct u_i, u_j from an infinite set $U = \{u_1, u_2, \dots\}$ has its own 'r-root' $\rho_{u_iu_j}$. Clearly, $\mathcal{I}^*$ is a model of $\mathcal{T}$, and so there exists $\mathcal{J}^* =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}^*$ such that $\mathcal{J}^*$ is a model of the $\mathcal{ALC}$ TBox $\mathcal{T}'$. Now, since U is infinite, there exist two $u_i, u_j \in U$ that are instances in $\mathcal{J}^*$ of the same concept names in $\mathcal{T}'$. The restriction of $\mathcal{J}^*$ to $\{\rho_{u_i,u_j}, u_i, u_j\}$ is a model of $\mathcal{T}'$, which is $\mathsf{sig}(\mathcal{T}')$-bisimilar to the restriction $\mathcal{I}'$ of $\mathcal{J}^*$ to $\{\rho_{u_i,u_j}, u_i\}$. Thus $\mathcal{I}'$ is a model of $\mathcal{T}$ satisfying $(*)$.

We now come to the detailed proof of Theorem 7. A *path* p in an interpretation $\mathcal{I}$ is a word $d_0r_0 \cdots r_{n-1}d_n$ such that $d_i \in \Delta^{\mathcal{I}}$, $r_i \in \mathsf{N_R}$, and $(d_i, d_{i+1}) \in r^{\mathcal{I}}$ for all $i < n$. By $\mathsf{tail}(p)$ we denote the final element of p. If $\mathcal{I}$ is a ditree interpretation, then for every $d \in \Delta^{\mathcal{I}}$ there exists a unique path p starting from the root $\rho_{\mathcal{I}}$ of p such that $\mathsf{tail}(p) = d$. We denote this path by $p^{\mathcal{I}}(d)$. Let $\mathcal{I}$ and $\mathcal{J}$ be ditree interpretations. A global bisimulation S between $\mathcal{I}$ and $\mathcal{J}$ is a *level bisimulation* if $(d, d') \in S$ implies that the length of $p^{\mathcal{I}}(d)$ equals the length of $p^{\mathcal{I}}(d')$. For $d \in \Delta^{\mathcal{I}}$ we denote by $\mathcal{I}_d$ the interpretation rooted at d.

Lemma 9 *Let $\mathcal{I}$ and $\mathcal{J}$ be globally bisimilar interpretations such that $\mathcal{I}$ is a model of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ and $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there are ditree interpretations $\mathcal{I}'$ and $\mathcal{J}'$ such that $\mathcal{I}'$ is a model of $\mathcal{T}$, $\mathcal{J}'$ is not a model of $\mathcal{T}$ and there is a level bisimulation S between $\mathcal{I}'$ and $\mathcal{J}'$. Moreover, we can assume that the outdegrees of $\mathcal{I}^*$ and $\mathcal{J}^*$ are finite.*

Proof. Assume $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ are globally bisimilar and $e \in C^{\mathcal{J}} \setminus D^{\mathcal{J}}$ for some $C \sqsubseteq D \in \mathcal{T}$. We unfold $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ to $\mathcal{I}^*$ and $\mathcal{J}^*$ as follows:

- $\Delta^{\mathcal{I}^*}$ is the set of all paths in $\mathcal{I}$ starting at d;
- $p \in A^{\mathcal{I}^*}$ if $\mathsf{tail}(p) \in A^{\mathcal{I}}$;
- $(p, p \cdot r \cdot f) \in r^{\mathcal{I}^*}$ if $(\mathsf{tail}(p), f) \in r^{\mathcal{I}}$.

$\mathcal{J}^*$ is defined analogously with paths in $\mathcal{J}$ starting at e. It is readily checked that $\mathcal{I}^*$ and $\mathcal{J}^*$ satisfy the conditions of the lemma except the bound on the outdegree. Let S be the level bisimulation.

We now define subinterpretations of $\mathcal{I}^*$ and $\mathcal{J}^*$ that have finite outdegree. The construction is by selective filtrations. We construct pairs (X, Y), where $X \subseteq \Delta^{\mathcal{I}^*}$ and $Y \subseteq \Delta^{\mathcal{J}^*}$.

- We start with $X = \{\rho_{\mathcal{I}^*}\}$ and $Y = \{\rho_{\mathcal{J}^*}\}$;
- Assume (X, Y) has been defined. Let $(d, e) \in S$ with $d \in X$ and $e \in Y$ such that no successor of d is in X. We find a subseteq X' of the set of successors of d with $|X'| \leq m_r$ such that whenever $d \in (\geqslant n\ r\ C)^{\mathcal{I}^*}$ and $(\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$, then there are at least n r successors of d in $C^{\mathcal{I}^*}$. Similarly we find such a set Y' of successors of e. Choose for every $d' \in X'$ with $(d, d') \in r^{\mathcal{I}^*}$ an e' with $(e, e') \in r^{\mathcal{J}^*}$ such that $(d', e') \in S$ and insert it into Y''. Also, choose for every $e' \in Y'$ with $(e, e') \in r^{\mathcal{J}^*}$ a d' with $(d, d') \in r^{\mathcal{I}^*}$ such that $(d', e') \in S$ and insert it into X''. Now set $X := X \cup X' \cup X''$ and $Y = Y \cup Y' \cup Y''$.

Let $\mathcal{I}'$ be the restriction of $\mathcal{I}^*$ to X and $\mathcal{J}'$ be the restriction of $\mathcal{J}^*$. It is readily checked that $\mathcal{I}', \mathcal{J}'$ are as required. ❑

Theorem 18 *An $\mathcal{ALCQ}$-TBox $\mathcal{T}$ is model-projectively $\mathcal{ALC}$-rewritable iff it is preserved under global bisimulations (and thus iff it is equivalent to an $\mathcal{ALC}$ TBox).*

Proof. Assume $\mathcal{T}$ is not preserved under global bisimulations and $\mathcal{T}'$ is a model-projective $\mathcal{ALC}$ rewriting of $\mathcal{T}$. By Lemma 9, there exist ditree interpretations $\mathcal{I}$ and $\mathcal{J}$, both of finite outdegree, such that $\mathcal{I}$ is a model of $\mathcal{T}$, $\mathcal{J}$ is not a model of $\mathcal{T}$, and there is a level bisimulation S between $\mathcal{I}$ and $\mathcal{J}$. We first show the following

Claim 1. There exists a ditree interpretation $\mathcal{I}'$ that is a model of $\mathcal{T}$ and is globally bisimilar to $\mathcal{I}$ such that for any two bisimilar $(\mathcal{I}', d)$ and $(\mathcal{I}', d')$ with d, d' points at the same level in $\mathcal{I}'$ the interpretations $\mathcal{I}'_d$ and $\mathcal{I}'_{d'}$ are isomorphic.

Proof of Claim 1. We define a sequence $\mathcal{I}_0, \mathcal{I}_1, \dots$ of ditree interpretations as follows:

- $\mathcal{I}_0 := \mathcal{I}$;
- Assume $\mathcal{I}_n$ has been defined. Let $\sim_n$ be the minimal bisimulation relation on points of level n in $\mathcal{I}_n$. For any equivalence class $[d]_{\sim_n} = \{d_1, \dots, d_m\}$ with respect to $\sim_n$ and any role name r in $\mathsf{sig}_R(\mathcal{T})$, take m disjoint copies $\mathcal{I}_e^1, \dots, \mathcal{I}_e^m$ of every $\mathcal{I}_e$ with e an r-successor of some $d_j \in [d]_{\sim_n}$ and attach $\mathcal{I}_e^i$ to d_i, for all $1 \leq i \leq m$, by connecting d_i and the the root of $\mathcal{I}_e^i$ using r. We assume $\mathcal{I}_e^j = \mathcal{I}_e$. Let $\mathcal{I}_{n+1}$ be the resulting interpretation.

Define $\mathcal{I}'$ as the union of all $\mathcal{I}_n$ (note that $\mathcal{I}_n$ is a subinterpretation of $\mathcal{I}_{n+1}$ since we assume $\mathcal{I}_e^j = \mathcal{I}_e$). It is readily checked that that for any two bisimilar $(\mathcal{I}', d)$ and $(\mathcal{I}', d')$ with d, d' points at the same level in $\mathcal{I}'$ the interpretations $\mathcal{I}'_d$ and $\mathcal{I}'_{d'}$ are isomorphic.

To prove that $\mathcal{I}'$ is a model of $\mathcal{T}$ observe that there is a p-morphism f from $\mathcal{I}'$ to $\mathcal{I}$. Since $\mathcal{T}'$ is a model conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$, there exists a model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. Thus, by Lemma 1, there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ with $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$. Thus $\mathcal{I}'$ is a model of $\mathcal{T}$. This finishes the proof of Claim 1.

Claim 2. There exists a ditree interpretation $\mathcal{I}''$ which is a model of $\mathcal{T}$ and is globally bisimilar to $\mathcal{I}'$ such that there do not exist two distinct globally bisimilar $\mathcal{I}_{d_1}$ and $\mathcal{I}_{d_2}$ with d_1, d_2 r-successors of some d in $\mathcal{I}''$.

We construct $\mathcal{I}''$ as the limit of a sequence $\mathcal{I}'_0, \mathcal{I}'_1, \dots$ defined as follows:

- $\mathcal{I}'_0 := \mathcal{I}'$;
- Assume $\mathcal{I}'_n$ has been defined. Consider a lowest level occurrence of *distinct* globally bisimilar $\mathcal{I}_{d_1}$ and $\mathcal{I}_{d_2}$ with d_1, d_2 r-successors of some d in $\mathcal{I}'_n$. (If this situation does not occur, set $\mathcal{I}'_{n+1} := \mathcal{I}'_n$.) Take such a d with r-successors $d_1, \dots, d_m$, $m > 1$, such that $\mathcal{I}_{d_1}, \dots, \mathcal{I}_{d_m}$ are globally bisimilar. By Claim 1 $\mathcal{I}_{d_1}, \dots, \mathcal{I}_{d_m}$ are isomorphic. We define $\mathcal{I}'_{n+1}$ as the result of removing $\mathcal{I}_{d_2}, \dots, \mathcal{I}_{d_m}$ from $\mathcal{I}'_n$.

We show that if $\mathcal{I}'_n$ is a model of $\mathcal{T}$, then $\mathcal{I}'_{n+1}$ is a model of $\mathcal{T}$.

Let U be a set of cardinality $\kappa > 2^{\aleph_0}$ and take for every $u \in U$ a copy $\mathcal{I}_u$ of the interpretation $\mathcal{I}_{d_1}$. We assume that the $\mathcal{I}_u$, $u \in U$, are mutually disjoint. For any m-element subset $W = \{w_1, \dots, w_m\}$ of U define an interpretation $\mathcal{I}_W$ that is obtained from $\mathcal{I}'_n$ by replacing the subinterpretations $\mathcal{I}_{d_1}, \dots, \mathcal{I}_{d_m}$ by $\mathcal{I}_{w_1}, \dots, \mathcal{I}_{w_m}$, respectively. We assume that the $\mathcal{I}_W$ are mutually disjoint except for the nodes in $\Delta^{\mathcal{I}_u}$, $u \in U$. Note that all $\mathcal{I}_W$ are isomorphic to $\mathcal{I}'_n$. Let $\mathcal{J}$ be the union of all $\mathcal{I}_W$. The point generated subinterpretations of $\mathcal{J}$ are all isomorphic to generated subinterpretations of $\mathcal{I}'_n$. Thus $\mathcal{J}$ is a model of $\mathcal{T}$ since $\mathcal{I}'_n$ is a model of $\mathcal{T}$. Hence there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. As U has cardinality $> 2^{\aleph_0}$, there is a set $W_0 = \{w_1, \dots, w_m\} \subseteq U$ of cardinality m such that the restrictions of $\mathcal{J}'$ to $\Delta^{\mathcal{I}_{w_i}}$ are isomorphic, for all $w_i \in W_0$. Let $\mathcal{I}'_{W_0}$ be the restriction of $\mathcal{J}'$ to $\Delta^{\mathcal{I}_{W_0}}$. The resulting interpretation $\mathcal{I}''_{W_0}$ after removing all points in $\mathcal{I}_{w_2}, \dots, \mathcal{I}_{w_m}$ from $\mathcal{I}'_{W_0}$ is clearly again a model of $\mathcal{T}'$ and $\mathcal{I}''_{W_0} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}''_n$. Thus $\mathcal{I}''_n$ is a model of $\mathcal{T}$.

Define $\mathcal{I}''$ as the limit of the sequence $\mathcal{I}'_0, \mathcal{I}'_1, \dots$. It is readily checked that $\mathcal{I}''$ is as required. This finishes the proof of Claim 2.

The interpretation $\mathcal{I}''$ obtained in Claim 2 is globally bisimilar to $\mathcal{J}$. So we have a level bisimulation S between $\mathcal{J}$ and $\mathcal{I}''$. From the condition for $\mathcal{I}''$ that no node has distinct bisimilar r-successors we obtain that S is a function, thus a p-morphism. By Lemma 1 there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J}' =_\Sigma \mathcal{J}$. Thus, $\mathcal{J}$ is a model of $\mathcal{T}$ and we have derived a contradiction. ❏

E.2 Proofs of 2ExpTime upper bounds for $\mathcal{ALCQ}$ to $\mathcal{ALC}$-rewritability

We show that for $\mathcal{ALCQ}$ TBoxes all three types of rewritability into $\mathcal{ALC}$ TBoxes can be decided in 2ExpTime. This proves Theorem 6 and shows the claim that equivalent and model-conservative $\mathcal{ALCQ}$ to $\mathcal{ALC}$-rewritability can be decided in 2ExpTime.

Theorem 19 *For $\mathcal{ALCQ}$ TBoxes the following holds: equivalent $\mathcal{ALC}$-rewritability,* m*-conservative $\mathcal{ALC}$-rewritability, and* s*-conservative $\mathcal{ALC}$-rewritability are decidable in 2*ExpTime.

Proof. We employ the model-theoretic criteria and use type elimination procedures.

First we show that it is decidable in 2ExpTime whether a $\mathcal{ALCQ}$ TBox is preserved under global bisimulations. Assume a $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is given. The 2ExpTime algorithm deciding preservation under global bisimulations is as follows. Consider the set tp of all types over $\mathsf{sub}(\mathcal{T})$ and its subset $\mathsf{tp}(\mathcal{T})$ of all types in tp that are satisfiable w.r.t. $\mathcal{T}$. The following rules are applied recursively to the set $\mathcal{E}$ of elements of $2^{\mathsf{tp}} \times 2^{\mathsf{tp}(\mathcal{T})}$:

(A) Remove (T, T') from $\mathcal{E}$ if not all $t \in T \cup T'$ contain the same concept names.

(EX) Remove (T, T') from $\mathcal{E}$ if there is a role name r such that there are no interpretations $\mathcal{I}_t, t \in T \cup T'$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that
  - all $\mathcal{I}_t, t \in T'$, are models of $\mathcal{T}$;
  - d_t satisfies t, for all $t \in T \cup T'$;
  - for each $t_0 \in T \cup T'$ and $(d_{t_0}, e_{t_0}) \in r^{\mathcal{I}_{t_0}}$ there exist $(d_t, e_t) \in r^{\mathcal{I}_t}$ for $t \in (T \cup T') \setminus \{t_0\}$, such that there exists $(S, S') \in \mathcal{E}$ with S the set of types realized by the nodes e_t, $t \in T$, and S' the set of types realized by the nodels e_t, $t \in T'$.

Denote by $\mathcal{E}_0$ the remaining set. One can show that $\mathcal{E}_0$ is the set of all (T, T') such that there exist models $\mathcal{I}_t$, $t \in T \cup T'$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that

- all $\mathcal{I}_t, t \in T'$, are models of $\mathcal{T}$;
- d_t satisfies t, for all $t \in T \cup T'$;
- all $(\mathcal{I}_t, d_t)$, $t \in T \cup T'$, are bisimilar.

It follows that $\mathcal{T}$ is not preserved under global bisimulations iff there exists $(\{t\}, \{t'\}) \in \mathcal{E}_0$ such that $t \notin \mathsf{tp}(\mathcal{T})$.

Now we show that it is decidable in 2ExpTime whether an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is preserved under inverse $\mathsf{sig}(\mathcal{T})$-morphisms. Assume an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is given. The 2ExpTime algorithm is as follows. The following rules are applied recursively to the set $\mathcal{E}$ of all elements of $2^{\mathsf{tp}} \times \mathsf{tp}(\mathcal{T})$:

(A) Remove (T, s) from $\mathcal{E}$ if not all $t \in T \cup \{s\}$ contain the same concept names.

(EX) Remove (T, s) from $\mathcal{E}$ if there is a role name r such that there are no interpretations $\mathcal{I}_t, t \in T \cup \{s\}$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that
  - $\mathcal{I}_s$ is a model of $\mathcal{T}$;
  - d_t satisfies t, for all $t \in T \cup \{s\}$;
  - for each $t \in T$ there is a function f_t from the set of $r^{\mathcal{I}_t}$-successors of d_t onto the set of $r^{\mathcal{I}_s}$-successor of d_s such that for each $r^{\mathcal{I}_s}$-successor e_s of d_s there exists $(S, s') \in \mathcal{E}$ such that s' is the type of e_s and S is the set of types realized in $\bigcup_{t \in T} f_t^{-1}(e_s)$.

Denote by $\mathcal{E}_0$ the remaining set. One can show that $\mathcal{E}_0$ is the set of all (T, s) such that there exist models $\mathcal{I}_t$, $t \in T \cup \{s\}$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that

- $\mathcal{I}_s$ is a model of $\mathcal{T}$;
- d_t satisfies t, for all $t \in T \cup \{s\}$;
- there are p-morphisms f_t from each $\mathcal{I}_t$ onto $\mathcal{I}_s$ with $f_t(d_t) = d_s$.

It follows that $\mathcal{T}$ is not preserved under inverse p-morphisms iff there exists $(\{t\}, s) \in \mathcal{E}_0$ such that $t \notin \mathsf{tp}(\mathcal{T})$. ❏

F $\mathcal{ALCQI}$ to $\mathcal{ALCI}$ rewritability

Given a $\mathcal{ALCQI}$ TBox $\mathcal{T}$, the construction of the $\mathcal{ALCI}$ TBox $\mathcal{T}^\dagger$ extends the definition of $\mathcal{T}^\dagger$ from the analysis of $\mathcal{ALCQ}$ to $\mathcal{ALC}$ rewritability by introducing concept names B_D for qualified number respiction D with inverse roles as well. No other changes are required. We aim to prove the following

Theorem 20 *Let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox. Then the following conditions are equivalent:*

1. *$\mathcal{T}$ is* s*-conservatively $\mathcal{ALCI}$-rewritable;*
2. *$\mathcal{T}$ is preserved under inverse i-p-morphisms.*
3. *$\mathcal{T}^\dagger$ is a* s*-conservative rewriting of $\mathcal{T}$.*

The proof is given by a sequence of lemmas. First observe that we have the following fundamental property of i-p-morphisms.

Lemma 10 *Suppose $\mathcal{T}$ is an $\mathcal{ALCI}$ TBox, Σ contains all role names in* $\mathsf{sig}(\mathcal{T})$*, and there is a Σ-i-p-morphism f from an interpretation $\mathcal{I}$ to some model $\mathcal{I}'$ of $\mathcal{T}$. Then there is a model $\mathcal{J}$ of $\mathcal{T}$ such that $\mathcal{J} =_\Sigma \mathcal{I}$.*

The following lemma proves the direction (2) $\Rightarrow$ (3) of Theorem 20.

Lemma 11 *If $\mathcal{T}$ is preserved under inverse i-p-morphisms, then $\mathcal{T}^\dagger$ is a* s*-conservative $\mathcal{ALCQ}$-rewriting of $\mathcal{T}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{T}^\dagger$ that is not a model of $\mathcal{T}$. We may assume that $\mathcal{I}$ is ω-saturated. Define an equivalence relation $\sim$ on $\Delta^\mathcal{I}$ by setting $d \sim d'$ iff $(\mathcal{I}, d)$ is Σ-i-bisimilar to $(\mathcal{I}, d')$. Denote by $[d]$ the equivalence class of $d \in \Delta^\mathcal{I}$. Define an interpretation $\mathcal{J}$ as follows:

- $\Delta^\mathcal{J} = \{[d] \mid d \in \Delta^\mathcal{I}\}$;
- $[d] \in A^\mathcal{J}$ iff $d \in A^\mathcal{I}$;
- $([d], [d']) \in r^\mathcal{J}$ iff there are $e \in [d]$ and $e' \in [d']$ such that $(e, e') \in r^\mathcal{I}$.

One can show that f is a Σ-i-p-morphism from $\mathcal{I}$ to $\mathcal{J}$. Thus, $\mathcal{J}$ is a model of $\mathcal{T}^\dagger$ that satisfies each $\mathcal{ALCI}_\Sigma$-type $\mathbf{t}$ at most once. The following can now we proved in a straightforward way using the construction of $\mathcal{T}^\dagger$.

Claim. If a model of $\mathcal{T}^\dagger$ satisfies each $\mathcal{ALCI}_\Sigma$-type $\mathbf{t}$ at most once, then it is a model of $\mathcal{T}$.

It follows that $\mathcal{J}$ is a model of $\mathcal{T}$ and, therefore, by preservation of $\mathcal{T}$ under inverse i-p-morphisms $\mathcal{I}$ is a model of $\mathcal{T}$. We have obtained a contradiction. ❑

We note the following analogue of Lemma 7.

Lemma 12 *An $\mathcal{ALCI}$ TBox $\mathcal{T}'$ is a* s*-conservative rewriting of an $\mathcal{ALCQI}$ TBox $\mathcal{T}$ iff $\mathcal{T}'$ is a* m*-conservative rewriting of $\mathcal{T}$ over the class of tree-interpretations that are counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r (in particular over the class of tree-interpretations of bounded outdegree).*

Proof. The proof of ($\Leftarrow$) follows immediately from the fact that if $\mathcal{T} \not\models C \sqsubseteq D$ for an $\mathcal{ALCQI}$ TBox $\mathcal{T}$ and an $\mathcal{ALCQI}$-CI $C \sqsubseteq D$, then there exists a tree interpretation that is counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r that is a model of $\mathcal{T}$ but refutes $C \sqsubseteq D$. The proof of ($\Rightarrow$) is similar to the proof of Lemma 7. Let $\mathcal{I}$ be a tree-interpretation that is counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r and is a model of $\mathcal{T}$. Let d be a root of $\mathcal{I}$. The type $t_\mathcal{I}^{\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}}(d)$ is satisfiable relative to $\mathcal{T}'$. Thus we find an ω-saturated model $\mathcal{J}$ of $\mathcal{T}'$ such that $\mathcal{I}$ is a subinterpretation of $\mathcal{J}$ and for each $d' \in \Delta^\mathcal{I}$, $t_\mathcal{I}^{\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}}(d) = t_\mathcal{J}^{\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}}(d)$. We can assume (by unfolding) that $\mathcal{J}$ is a tree-interpretation as well. Now one can use selective filtration over the subconcepts of $\mathcal{T}'$ to construct a model $\mathcal{J}'$ of $\mathcal{T}'$ with $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. ❑

The following lemma proves the direction (1) $\Rightarrow$ (2) of Theorem 20.

Lemma 13 *Let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox. If $\mathcal{T}$ is* s*-conservatively $\mathcal{ALCI}$-rewritable, then $\mathcal{T}$ is preserved under inverse i-p-morphisms.*

Proof. Suppose an $\mathcal{ALCQI}$ TBox $\mathcal{T}$ is s-conservatively $\mathcal{ALCI}$-rewritable. Let $\mathcal{T}'$ be an s-conservative $\mathcal{ALCI}$-rewriting of $\mathcal{T}$. We may assume that $\mathcal{T}'$ does not use fresh role names. For a proof by contradiction assume that there are interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ and an i-p-morphism f from $\mathcal{I}_1$ to $\mathcal{I}_2$ such that $\mathcal{I}_2 \models \mathcal{T}$ and $\mathcal{I}_1 \not\models \mathcal{T}$. We may assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ are both ω-saturated. Let $\mathcal{I}_1^\dagger$ and $\mathcal{I}_2^\dagger$ be the unfoldings of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Note that $\mathcal{I}_i \models \mathcal{T}$ iff $\mathcal{I}_i^\dagger \models \mathcal{T}$ and that we can lift f to an i-p-morphism $f^\dagger$ from $\mathcal{I}_1^\dagger$ to $\mathcal{I}_2^\dagger$. We therefore obtain a contradiction if $\mathcal{I}_1^\dagger \models \mathcal{T}$. By Lemma 12 there exists a model $\mathcal{J}_2$ of $\mathcal{T}'$ such that $\mathcal{J}_2 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_2^\dagger$. But then, by Lemma 10 there exists a model $\mathcal{J}_1$ of $\mathcal{T}'$ with $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1^\dagger$. Hence $\mathcal{I}_1^\dagger \models \mathcal{T}$. ❑

We now prove the claim from Example 6.

Lemma 14 *Let*

$$\mathcal{T} = \{\exists r.\top \sqsubseteq \exists r.(\geq\ 2r^-.\top)\}.$$

Then $\mathcal{T}' = \{\exists r.\top \sqsubseteq \exists r.(\exists r^-.B \sqcap \exists r^-.\neg B)\}$ is a model-conservative $\mathcal{ALCI}$-rewriting of $\mathcal{T}$.

Proof. Clearly, $\mathcal{T}' \models \mathcal{T}$. Conversely, assume $\mathcal{I} \models \mathcal{T}$. We have to show that there exists a model $\mathcal{J}$ of $\mathcal{T}'$ such that $\mathcal{J} =_{\{r\}} \mathcal{I}$. We may assume that $B^\mathcal{I} = \emptyset$. We use transifnite recursion to construct a sequence $(\mathcal{J}_\alpha, m_\alpha)$ of interpretations $\mathcal{J}_\alpha$ and sets $m_\alpha \subseteq \Delta^\mathcal{I}$ of the elements of $(\exists r.\top)^\mathcal{I}$ as follows:

Zero case ($\alpha = 0$) set $\mathcal{J}_0 = \mathcal{I}$ and $m_0 = \emptyset$.

Assume now all $(\mathcal{J}_\gamma, m_\gamma)$ with $\gamma < \alpha$ are defined.

Limit case (α is a limit ordinal) set

$$\mathcal{J}_\alpha = \bigcup_{\gamma<\alpha} \mathcal{J}_\gamma, \quad m_\alpha = \bigcup_{\alpha<\gamma} m_\alpha$$

Successor case (α is a successor ordinal) let $\alpha = \gamma_0 + 1$. If $d \in m_\alpha$ for all $d \in (\exists r.\top)^{\mathcal{J}_{\gamma_0}}$, set $\mathcal{J}_\alpha = \mathcal{J}_{\gamma_0}$ and $m_\alpha = m_{\gamma_0}$.

Otherwise choose $d_0 \in (\exists r.\top)^\mathcal{I}$ with $d_0 \notin m_{\gamma_0}$. Let e and $d_1 \neq d_0$ be such that $(d_0, e) \in r^\mathcal{I}$ and $(d_1, e) \in r^\mathcal{I}$. Consider cases.

(a) $d_1 \in m_{\gamma_0}$ and $d_1 \notin B^{\mathcal{J}_{\gamma_0}}$. Then set

$$B^{\mathcal{J}_\alpha} = B^{\mathcal{J}_{\gamma_0}} \cup \{d_0\}, \quad m_\alpha = m_{\gamma_0} \cup \{d_0\}.$$

(b) case (a) above does not hold but $d_1 \in m_{\gamma_0}$ and $d_1 \in B^{\mathcal{J}_{\gamma_0}}$. Then set

$$B^{\mathcal{J}_\alpha} = B^{\mathcal{J}_{\gamma_0}}, \quad m_\alpha = m_{\gamma_0} \cup \{d_0\}.$$

(c) Otherwise, $d_1 \notin m_{\gamma_0}$. In this case set

$$B^{\mathcal{J}_\alpha} = B^{\mathcal{J}_{\gamma_0}} \cup \{d_0\}, \quad m_\alpha = m_{\gamma_0} \cup \{d_0, d_1\}.$$

Clearly the sequence $(\mathcal{J}_\alpha, m_\alpha)$ stabilizes at some ordinal β. We set $\mathcal{J} = \mathcal{J}_\beta$. It is readily checked that $\mathcal{J}$ is a model of $\mathcal{T}'$. ❑

G $\mathcal{ALCQI}$ to $\mathcal{ALC}$ rewritability

Theorem 8 Model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability relative to a signature Σ is undecidable.

Proof. It is known [Konev et al., 2013] that there exists an $\mathcal{ALC}$ TBox of the form $\mathcal{T}_0 = \{\top \sqsubseteq \exists r.C\}$ and a signature Σ_0 not containing r such that it is undecidable whether for every interpretation $\mathcal{I}$ there exists a model $\mathcal{J}$ of $\mathcal{T}_0$ with $\mathcal{J} =_{\Sigma_0} \mathcal{I}$ (in symbols $\emptyset \equiv_{\Sigma_0} \mathcal{T}_0$). Consider $\mathcal{T} = \{\top \sqsubseteq \exists s^-.\top \sqcup \exists r.C\}$, where s is fresh, and set $\Sigma = \Sigma_0 \cup \{s\}$. We show that $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable w.r.t. Σ iff $\emptyset \equiv_{\Sigma_0} \mathcal{T}_0$: if $\emptyset \equiv_{\Sigma_0} \mathcal{T}_0$, then the empty TBox is a model-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$. If $\emptyset \not\equiv_{\Sigma_0} \mathcal{T}_0$, then take an interpretation $\mathcal{I}$ for which there does not exist a model $\mathcal{J}$ of $\mathcal{T}_0$ with $\mathcal{J} =_{\Sigma_0} \mathcal{I}$. Pick $d_0 \in \Delta^{\mathcal{I}}$ and assume w.l.o.g. that $s^{\mathcal{I}} = \{d_0\} \times \Delta^{\mathcal{I}}$. Clearly $\mathcal{I}$ is a model of $\mathcal{T}$. Assume there exists an $\mathcal{ALC}$ TBox $\mathcal{T}'$ that is a model-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$ w.r.t. Σ. Let $\mathcal{J}$ be a model of $\mathcal{T}'$ such that $\mathcal{J} =_\Sigma \mathcal{I}$. We may assume that $|\Delta^{\mathcal{I}}| > 2^{|\mathcal{T}'|}$. Now remove (d_0, d) from $s^{\mathcal{J}}$ for some $d \neq d_0$ such that the $\mathcal{T}'$-type of d is satisfied in some $d' \neq d$. The resulting interpretation is still a model of $\mathcal{T}'$ (since $\mathcal{T}'$ is an $\mathcal{ALC}$ TBox) but it is not a model of $\mathcal{T}$. We have derived a contradiction. ❑

Our aim now is prove Theorem 9.

Let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox. We take the $\mathcal{ALCQ}$-rewriting $\mathcal{T}'$ of $\mathcal{T}$ and then apply the $\mathcal{ALC}$ rewriting $\cdot^\dagger$ to $\mathcal{T}'$. In detail, let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox and assume that $\mathcal{T}$ is built using $\neg$, $\sqcap$ and $(\geqslant n\ r\ C)$ only. Set

$$\mathcal{T}' = \{C^\uparrow \sqsubseteq D^\uparrow \mid \mathcal{T} \models C \sqsubseteq D\}$$

where $C^\uparrow$ replaces the top-most occurrences of qualified number restrictions $D = (\geq\ n\ r\ C)$ with inverse roles r by fresh concept names B_D. Take fresh concept names $B_D, B_1^D, \ldots, B_n^D$, for $D = (\geqslant\ n\ r\ C) \in \mathsf{sub}(\mathcal{T}')$, and let Σ be $\mathsf{sig}(\mathcal{T}')$ together with the fresh concept names. For $C \in \mathsf{sub}(\mathcal{T}')$, let $C^\sharp$ be the $\mathcal{ALC}$-concept obtained from C by replacing all top-most occurrences of any $D = (\geqslant\ n\ r\ D')$ in C with B_D. Now, define $(\mathcal{T}')^\dagger$ to be the *infinite* TBox containing $C^\sharp \sqsubseteq D^\sharp$, for $C \sqsubseteq D \in \mathcal{T}'$, and for all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T}')$,

- $B_i^D \sqsubseteq \neg B_j^D$ for $i \neq j$,
- $B_D \sqsubseteq \exists r.(C^\sharp \sqcap B_1^D) \sqcap \cdots \sqcap \exists r.(C^\sharp \sqcap B_n^D)$,
- $\bigsqcap_{1 \leq i \leq n} (\exists r.(C^\sharp \sqcap C_i^\sharp \sqcap \bigsqcap_{j \neq i} \neg C_j^\sharp)) \sqsubseteq B_D$, for any $\mathcal{ALC}$-concepts C_i with $\mathsf{sig}(C_i) \subseteq \Sigma$.

Theorem 21 *Let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox. Then the following conditions are equivalent:*

1. *$\mathcal{T}$ is* s*-conservatively $\mathcal{ALC}$-rewritable;*
2. *$\mathcal{T}$ is preserved under inverse p-morphisms.*
3. *$(\mathcal{T}')^\dagger$ is an* s*-conservative rewriting of $\mathcal{T}$.*

The proof is given by a sequence of lemmas. The following lemma proves the direction (2) ⇒ (3) of Theorem 21.

Lemma 15 *If $\mathcal{T}$ is preserved under inverse p-morphisms, then $(\mathcal{T}')^\dagger$ is an* s*-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$.*

Proof. We show that $(\mathcal{T}')^\dagger \models \mathcal{T}$. The proof is by contraction. Let $\mathcal{I}$ be a model of $(\mathcal{T}')^\dagger$ that is not a model of $\mathcal{T}$. We may assume that $\mathcal{I}$ is ω-saturated. By the proof of Lemma 11 we obtain a model $\mathcal{I}'$ of $\mathcal{T}'$ that is a p-morphic image of $\mathcal{I}$. We obtain a contradiction if $\mathcal{I}'$ is a model of $\mathcal{T}$. But this follows from $\mathcal{T}' \models \mathcal{T}$ where the latter statement follows from the proof of Theorem 4 and the assumption that $\mathcal{T}$ is preserved under inverse p-morphims (which implies that $\mathcal{T}$ is preserved under inverse counting p-morphisms). ❑

The following lemma can be proved similarly to Lemma 12.

Lemma 16 *An $\mathcal{ALC}$ TBox $\mathcal{T}'$ is a* s*-conservative rewriting of an $\mathcal{ALCQI}$ TBox $\mathcal{T}$ iff $\mathcal{T}'$ is a* m*-conservative rewriting of $\mathcal{T}$ over the class of tree-interpretations that are counting $\mathcal{ALCQI}_{\mathsf{sig}(\mathcal{T})}$-saturated for every role r (in particular over the class of tree-interpretations of bounded outdegree).*

The following lemma proves the direction (1) ⇒ (2) of Theorem 21.

Lemma 17 *Let $\mathcal{T}$ be an $\mathcal{ALCQI}$ TBox. If $\mathcal{T}$ is* s*-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}$ is preserved under inverse p-morphisms.*

Proof. Suppose an $\mathcal{ALCQI}$ TBox $\mathcal{T}$ is s-conservatively $\mathcal{ALC}$-rewritable. Let $\mathcal{T}'$ be an s-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$. We may assume that $\mathcal{T}'$ does not use fresh role names. For a proof by contradiction assume that there are interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ and an p-morphism f from $\mathcal{I}_1$ to $\mathcal{I}_2$ such that $\mathcal{I}_2 \models \mathcal{T}$ and $\mathcal{I}_1 \not\models \mathcal{T}$. We may assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ are both ω-saturated. Let $\mathcal{I}_1^\dagger$ and $\mathcal{I}_2^\dagger$ be the unfoldings of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Note that $\mathcal{I}_i \models \mathcal{T}$ iff $\mathcal{I}_i^\dagger \models \mathcal{T}$ and that we can lift f to a p-morphism $f^\dagger$ from a disjoint union $\mathcal{I}_1'$ of copies of $\mathcal{I}_1^\dagger$ to $\mathcal{I}_2^\dagger$. We therefore obtain a contradiction if $\mathcal{I}_1' \models \mathcal{T}$. By Lemma 16 there exists a model $\mathcal{J}_2$ of $\mathcal{T}'$ such that $\mathcal{J}_2 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_2^\dagger$. But then, by Lemma 1 there exists a model $\mathcal{J}_1$ of $\mathcal{T}'$ with $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1'$. Hence $\mathcal{I}_1' \models \mathcal{T}$. ❑

H Proofs for DL-Lite$_{horn}$

Theorem 22 *An any $\mathcal{ALCI}$-TBox $\mathcal{T}$ the following conditions are equivalent:*

- *$\mathcal{T}$ is equivalently DL-Lite$_{horn}$-rewritable;*
- *$\mathcal{T}$ is* s*-conservatively DL-Lite$_{horn}$-rewritable;*
- *$\mathcal{T}$ is* m*-conservatively DL-Lite$_{horn}$-rewritable.*

Proof. Assume $\mathcal{T}'$ is an s-conservative *DL-Lite$_{horn}$*-rewriting of an $\mathcal{ALCI}$ TBox $\mathcal{T}$. By Theorem 1, we may assume that $\mathcal{T}'$ does not contain additional role names. Let $\mathcal{T}''$ be the set of *DL-Lite$_{horn}$*-inclusions $C \sqsubseteq D$ in $\mathsf{sig}(\mathcal{T})$ such that C, D do not contain redundant conjuncts (and so $\mathcal{T}''$ is finite) and $\mathcal{T}' \models C \sqsubseteq D$. It is sufficient to show that $\mathcal{T}''$ is an equivalent *DL-Lite$_{horn}$*-rewriting of $\mathcal{T}$. Clearly $\mathcal{T} \models \mathcal{T}''$. Thus, assume $\mathcal{T}'' \not\models \mathcal{T}$. Let $\mathcal{I}$ be a model of $\mathcal{T}''$ that is not a model of $\mathcal{T}$. We expand $\mathcal{I}$ to an interpretation $\mathcal{I}'$ by setting for any concept name $M \notin \mathsf{sig}(\mathcal{T})$ and $d \in \Delta^{\mathcal{I}}$, $d \in M^{\mathcal{I}'}$ iff $\mathcal{T}' \models D \sqsubseteq M$, where D is the conjunction of all basic *DL-Lite$_{horn}$* concepts B with $\mathsf{sig}(B) \subseteq \mathsf{sig}(\mathcal{T})$ and $d \in B^{\mathcal{I}}$. Then, since $\mathcal{T}'$ does not contain any additional role names, $\mathcal{I}'$ is a model of $\mathcal{T}'$. Thus $\mathcal{T}' \not\models \mathcal{T}$ and we have derived a contradiction. ❑