Robust, Expressive, and Quantitative Linear Temporal Logics: Pick any Two for Free[*]

Daniel Neider[1], Alexander Weinert[2] and Martin Zimmermann[3]

[1] Max Planck Institute for Software Systems, 67663 Kaiserslautern, Germany
`neider@mpi-sws.org`
[2] German Aerospace Center (DLR), 51147 Cologne, Germany
`alexander.weinert@dlr.de`
University of Liverpool, Liverpool L69 3BX, United Kingdom
`martin.zimmermann@liverpool.ac.uk`

Abstract. Linear Temporal Logic (LTL) is the standard specification language for reactive systems and is successfully applied in industrial settings. However, many shortcomings of LTL have been identified in the literature, among them the limited expressiveness, the lack of quantitative features, and the inability to express robustness. There is work on overcoming these shortcomings, but each of these is typically addressed in isolation. This is insufficient for applications where all shortcomings manifest themselves simultaneously.

Here, we tackle this issue by introducing logics that address more than one shortcoming. To this end, we combine the logics Linear Dynamic Logic, Prompt-LTL, and robust LTL, each addressing one aspect, to new logics. For all combinations of two aspects, the resulting logic has the same desirable algorithmic properties as plain LTL. In particular, the highly efficient algorithmic backends that have been developed for LTL are also applicable to these new logics. Finally, we discuss how to address all three aspects simultaneously.

1 Introduction

Linear Temporal Logic (LTL) [21] is amongst the most prominent and most important specification languages for reactive systems, e.g., non-terminating controllers interacting with an antagonistic environment. Verification of such systems against LTL specifications is routinely applied in industrial settings nowadays [11,14]. Underlying this success story is the exponential compilation property [28]: every LTL formula can be effectively translated into an equivalent Büchi automaton of exponential size (and it turns out that this upper bound is tight). In fact, almost all verification algorithms for LTL are based on this property, which is in particular true for the popular polynomial space model checking algorithm and the doubly-exponential time synthesis algorithms. Other desirable properties of LTL include its compact and variable-free syntax and its intuitive semantics.

Despite the success of LTL, a plethora of extensions of LTL have been studied, all addressing individual and specific shortcomings of LTL, e.g., its limited expressiveness, its lack of quantitative features, and its inability to express robustness. Commonly, extensions of LTL as described above are only studied in isolation—the logics are either more expressive, or quantitative, or robust. One notable exception is Parametric LDL (PLDL) [13], which adds quantitative operators and increased expressiveness while maintaining the exponential compilation property and intuitive syntax and semantics. In practical settings, however, it does not suffice to address one shortcoming of LTL while ignoring the others. Instead, one needs a logic that combines multiple extensions while still maintaining the desirable properties of LTL. The overall goal of this paper is, hence, to bridge this gap, thereby enabling expressive, quantitative, and robust verification and synthesis.

It is a well-known fact that LTL is strictly weaker than Büchi automata, i.e., it does not harness the full expressive power of the algorithmic backends. Thus, increasing the expressiveness of LTL has generated much attention [18,27,28,30] as it can be easily exploited: as long as the new logic also has the exponential compilation property, the same optimized backends as for LTL can be used. A prominent and recent example of such an extension that yields the full expressive power of Büchi automata is Linear Dynamic Logic (LDL) [27], which adds to LTL temporal operators guarded by regular expressions. As an

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example, consider the specification "p holds at every even time point, but may or may not hold at odd time points". It is well-known that this property is not expressible in LTL, as LTL, intuitively, is unable to count modulo a fixed number. However, the specification is easily expressible in LDL as $[r]\,p$, where r is the regular expression $(\texttt{tt} \cdot \texttt{tt})^*$. The formula requires p to be satisfied at every position j such that the prefix up to position j matches the regular expression r (which is equivalent to j being even), i.e., $\texttt{tt}$ is an atomic regular expression that matches every letter. In this work, we consider LDL instead of the alternatives cited above for its conceptual simplicity: LDL has a simple and variable-free syntax based on regular expressions as well as intuitive semantics (assuming some familiarity with regular expressions).

Another serious shortcoming of LTL (and LDL) is its inability to adequately express timing bounds. For example, consider the specification "every request q is eventually answered by a response p", which is expressed in LTL as $\Box(q \rightarrow \Diamond p)$. It is satisfied, even if the waiting time between requests q and responses p diverges to infinity, although such a behavior is typically undesired. Again, a long line of research has addressed this second shortcoming of LTL [1,13,16,17,32]. The most basic one is Prompt-LTL [17], which adds the prompt-eventually operator $\Diamond_{\mathbf{p}}$ to LTL. The semantics is now defined with an additional parameter k, which bounds the scope of $\Diamond_{\mathbf{p}}$: $\Box(q \rightarrow \Diamond_{\mathbf{p}} p)$ requires every request q to be answered within k steps, when evaluated with respect to k. The resulting logic is a quantitative one: either one quantifies the parameter k existentially and obtains a boundedness problem, e.g., "is there a bound k such that every request can be answered within k steps", or one even aims to determine the optimal bound k. Again, Prompt-LTL retains the desirable properties of LTL, i.e., the exponential compilation property as well as intuitive syntax and semantics. Furthermore, Prompt-LTL captures the technical core of the alternatives cited above, e.g., decision problems for the more general logic PLTL [1] can be reduced to those for Prompt-LTL. For these reasons, we study Prompt-LTL in this work.

Finally, a third line of extensions of LTL is concerned with the concept of robustness, which is much harder to formalize. This is reflected by a multitude of incomparable notions of robustness in verification [5,7,9,10,12,19,20,25,26]. Here, we are interested in robust LTL (rLTL) [26], which equips LTL with a five-valued semantics that captures different degrees of violations of universal specifications. As an example, consider the specification "if property φ always holds true, then property ψ also always holds true", which is expressed in LTL as $\Box\varphi \rightarrow \Box\psi$ and is typical for systems that have to interact with an antagonistic environment. In classical semantics, the whole formula is satisfied as soon as the assumption φ is violated once, even if the guarantee ψ is violated as well. By contrast, the semantics of robust LTL ensures that the degree of the violation of $\Box\psi$ is always proportional to the degree of the violation of $\Box\varphi$. To this end, the degree of a violation of a property $\Box\varphi$ is expressed by five different truth values: either φ always holds, or φ is violated only finitely often, violated infinitely often, violated almost always, or violated always. Again, robust LTL has the exponential compilation property and an intuitive syntax (though its semantics is more intricate). In this work, we consider robust LTL, as it is the first logic that intrinsically captures the notion of robustness in LTL. In particular, formulas of robust LTL are evaluated over traces with Boolean truth values for atomic propositions and do not require non-Boolean assignments, which are often hard to determine in real-life applications.

Our Contributions We develop logics that address more than one shortcoming of LTL at a time. See Figure 1 for an overview.

In Section 3, we "robustify" Prompt-LTL. More precisely, we introduce a novel logic, named rPrompt-LTL, by extending the five-valued semantics from robust LTL to Prompt-LTL. Our main result here shows that rPrompt-LTL retains the exponential complication property. Then, In Section 4, we "robustify" LDL: we introduce a novel logic, named rLDL, by lifting the five-valued semantics of robust LTL to LDL. Our main result shows that rLDL also retains the exponential compilation property. Hence, one can indeed combine any two of the three extensions of LTL while still preserving the desirable algorithmic properties of LTL. In particular, let us stress again that all highly sophisticated algorithmic backends developed for LT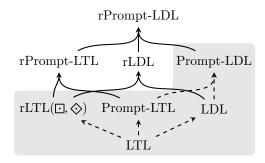L are applicable to these novel logics as well, e.g., we show that the verification problem and the synthesis problem for each of these logics is solvable without an (asymptotic) increase in complexity.

Fig. 1. The logics studied in this work. Existing logics and influences are marked gray with dashed arrows.

Tabuada and Neider gave two proofs showing that robust LTL has the exponential compilation property. The first one presented a translation of robust LTL into equivalent Büchi automata of exponential size while the second one is based on a polynomial translation of robust LTL into (standard) LTL, which is known to be translatable into equivalent Büchi automata of exponential size. We refer to those two approaches as the *direct* approach and the *reduction-based* approach. To obtain our results mentioned above, we need to generalize both. To prove the exponential compilation property for rLDL, we generalize the direct approach by exhibiting a direct translation of rLDL into Büchi automata via alternating automata. In contrast, to prove the exponential compilation property for rPrompt-LTL, we present a generalization of the reduction-based approach translating rPrompt-LTL into equivalent Prompt-LTL formulas of linear size, which have the exponential compilation property.

Finally, in Section 5, we discuss the combination of all three aspects. Recall that we present a direct translation to automata for rLDL and a reduction-based one for rPrompt-LTL. For reasons we discuss in Section 5, it is challenging to develop a reduction from rLDL to LDL or a direct translation for rPrompt-LTL that witness the exponential compilation property. Hence, both approaches seem inadequate to deal with the combination of all three extensions. Ultimately, we leave the question of whether the logic combining all three aspects has the exponential compilation property for future work.

Proofs omitted due to space restrictions can be found in the appendix.

2 Preliminaries

We denote the non-negative integers by $\mathbb{N}$, the set $\{0, 1\}$ of Boolean truth values by $\mathbb{B}$, and the power set of S by 2^S. By convention, we have $\min \emptyset = 1$ and $\max \emptyset = 0$ when the operators range over subsets of $\mathbb{B}$. Following Tabuada and Neider [26], the set of truth values for robust semantics is $\mathbb{B}_4 = \{0000, 0001, 0011, 0111, 1111\}$, which are ordered by $0000 \prec 0001 \prec 0011 \prec 0111 \prec 1111$. We write $\preceq$ for the non-strict variant of $\prec$ and define $\min \emptyset = 1111$ and $\max \emptyset = 0000$ when the operators range over subsets of $\mathbb{B}_4$.

Throughout this work, we fix a finite non-empty set P of atomic propositions. For a set $A \subseteq P$ and a propositional formula ϕ over P, we write $A \models \phi$ if the variable valuation mapping elements in A to 1 and elements in $P \setminus A$ to 0 satisfies ϕ. A trace (over P) is an infinite sequence $w \in (2^P)^\omega$. Given a trace $w = w(0)w(1)w(2)\cdots$ and a position $j \in \mathbb{N}$, we define $w[0, j) = w(0) \cdots w(j-1)$ and $w[j, \infty) = w(j)w(j+1)w(j+2)\cdots$, i.e., $w[0, j)$ is the prefix of length j of w and $w[j, \infty)$ the remaining suffix. In particular, $w[0, 0)$ is empty and $w[0, \infty)$ is w.

Our work is based on three logics, Robust Linear Temporal Logic (rLTL($\boxdot, \Diamonddot$)) [26], Linear Dynamic Logic (LDL) [27], and Prompt Linear Temporal Logic (Prompt-LTL) [17], which we briefly review in the in the following three subsections. More formal definitions can be found in the appendix and in the original publications introducing these logics.

We define the semantics of all these logics by evaluation functions V mapping a trace, a formula, and a bound (in the case of a quantitative logic) to a truth value. This is prudent for robust semantics, hence we also use this approach for the other logics, which are typically defined via satisfaction relations. In particular, V^{R}, V^{D}, and V^{P} denote the evaluation functions of rLTL($\boxdot, \Diamonddot$), LDL, and Prompt-LTL, respectively. Nevertheless, our definitions here are equivalent to the original definitions.

2.1 Robust Linear Temporal Logic

The main impetus behind the introduction of robust LTL was the need to capture the concept of robustness in temporal logics. As a first motivating example consider the LTL formula $\Box p$, stating that p holds at every position. Consequently, the formula is violated if there is a single position where p does not hold. However, this is a very "mild" violation of the property and there are much more "severe" violations. As exhibited by Tabuada and Neider, there are four canonical *degrees* of violation of $\Box p$: (i) p is violated at finitely many positions, (ii) p is violated at infinitely many positions, (iii) p is violated at all but finitely many positions, and (iv) p is violated at all positions. These first three degrees are captured by the LTL formulas $\Diamond \Box p$, $\Box \Diamond p$, and $\Diamond p$, which are all weakenings of $\Box p$. All five possibilities, satisfaction and four degrees of violation, are captured in robust LTL by the truth values

$$1111 \succ 0111 \succ 0011 \succ 0001 \succ 0000$$

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introduced above. By design, the formula $\boxdot p$ of robust LTL[3] has

- truth value 1111 on all traces where p holds at all positions,
- truth value 0111 on all traces where p holds at all but finitely many positions,
- truth value 0011 on all traces where p holds at infinitely many positions and does not hold at infinitely many positions,
- truth value 0001 on all traces where p only holds at finitely many positions, and
- truth value 0000 on all traces where p holds at no position.

As a further example, consider the formula $\boxdot p \rightarrow \boxdot q$. For this formula, the robust semantics captures the intuition described in the introduction: the implication is satisfied (i.e., has truth value 1111), if the degree of violation of the property "always q" is at most the degree of violation of the property "always p". Thus, if p is violated finitely often, then q may also be violated finitely often (but not infinitely often) while still satisfying the implication.

Conjunction and disjunction are defined as usual using minimization and maximization relying on the order indicated above while negation is based on the intuition that 1111 represents satisfaction and all other truth values represent degrees of violation. Hence, a negation $\neg \varphi$ is satisfied (i.e., has truth value 1111), if φ has truth value less than 1111, and it is violated (i.e., has truth value 0000) if φ has truth value 1111. Finally, the semantics of the eventually operator is defined as usual, i.e., the truth value of $\diamonddot \varphi$ on w is the maximal truth value that is assumed by φ on some suffix of w.

This intuition is formalized in the evaluation function V^{R} mapping a trace $w \in (2^P)^\omega$ and an rLTL$(\boxdot, \diamonddot)$ formula φ to a truth value $V^{\mathrm{R}}(w, \varphi)$ in $\mathbb{B}_4$. The formal definition is presented in Appendix A.1. Note that, for the sake of simplicity, we restrict ourselves to rLTL$(\boxdot, \diamonddot)$, i.e., robust LTL with the always and eventually operators, but without next, until, and release. We comment on the effect of this restriction when defining the combinations of logics.

2.2 Linear Dynamic Logic

The logic LDL has only two temporal operators, $\langle r \rangle$ and $[r]$, which can be understood as guarded variants of the classical eventually and always operators from LTL, respectively. Both are guarded by regular expressions r over the atomic propositions that may contain tests, which are again LDL formulas. These two operators together with Boolean connectives capture the full expressive power of the ω-regular expressions, i.e., LDL exceeds the expressiveness of LTL.

Formally, a formula $\langle r \rangle \varphi$ is satisfied by a trace w, if there is some j such that the prefix $w[0, j)$ matches the regular expression r and the corresponding suffix $w[j, \infty)$ satisfies φ. Dually, a formula $[r] \varphi$ is satisfied by a trace w if for every j with $w[0, j)$ matching r, $w[j, \infty)$ satisfies φ. Thus, while the classical eventually and always operator range over all positions, the operators of LDL range only over those positions whose induced prefix matches the guard of the operator.

The semantics of LDL is captured by the evaluation function V^{D} mapping a trace $w \in (2^P)^\omega$ and an LDL formula φ to a truth value $V^{\mathrm{D}}(w, \varphi)$ in $\mathbb{B}$. It is formally introduced in Appendix A.2.

2.3 Prompt Linear Temporal Logic

To express timing constraints, the logic Prompt-LTL adds the prompt-eventually operator $\diamond_{\mathbf{P}}$ to LTL. For technical reasons [1], this requires to disallow negation and implication. Intuitively, the new operator requires its argument to be satisfied within a bounded number of steps.

Thus, the semantics is given by an evaluation function V^{P} that maps a trace $w \in (2^P)^\omega$, a Prompt-LTL formula φ, and a bound $k \in \mathbb{N}$ to a truth value $V^{\mathrm{R}}(w, k, \varphi)$ in $\mathbb{B}$. This function is defined as usual for all standard Boolean and temporal operators, while a formula $\diamond_{\mathbf{P}} \varphi$ is satisfied with respect to the bound k if φ holds within the next k steps, i.e., the prompt-eventually behaves like the classical eventually with a bounded scope. The formal definition is presented in Appendix A.3.

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[3] Following the precedent for robust LTL, we use dots distinguish operators of robust logics from those of classical logics troughout the paper.

3 Robust and Prompt Linear Temporal Logic

We begin our treatment of combinations of the three basic logics by introducing robust semantics for Prompt-LTL, obtaining the logic rPrompt-LTL. To this end, we add the prompt-eventually operator to rLTL($\boxdot, \Diamonddown$) while disallowing implications and restricting negation to retain decidability (cf. [1]). The formulas of rPrompt-LTL are given by

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamonddown \varphi \mid \boxdot \varphi \mid \Diamonddown_{\mathbf{p}} \varphi,$$

where p ranges over the set P of atomic propositions. The size $|\varphi|$ of a formula φ is the number of its distinct subformulas.

The semantics of rPrompt-LTL is given by an evaluation function V^{RP} mapping a trace w, a bound k for the prompt-eventuallies, and a formula φ to a truth value in $\mathbb{B}_4$. To simplify our notation, we write $V_i^{\mathrm{RP}}(w, k, \varphi)$ for $i \in \{1, 2, 3, 4\}$ to denote the i-th bit of $V^{\mathrm{RP}}(w, k, \varphi)$, i.e.,

$$V^{\mathrm{RP}}(w, k, \varphi) = V_1^{\mathrm{RP}}(w, k, \varphi) V_2^{\mathrm{RP}}(w, k, \varphi) V_3^{\mathrm{RP}}(w, k, \varphi) V_4^{\mathrm{RP}}(w, k, \varphi).$$

The semantics of Boolean connectives as well as of the eventually and always operators is defined as for robust LTL. The motivation behind these definitions is carefully and convincingly discussed by Tabuada and Neider [26]. The semantics of the prompt-eventually operator bounds its scope to the next k positions as in classical Prompt-LTL [17].

- $V^{\mathrm{RD}}(w, k, p) = \begin{cases} 1111 & \text{if } p \in w(0), \\ 0000 & \text{if } p \notin w(0), \end{cases}$ • $V^{\mathrm{RD}}(w, k, \neg p) = \begin{cases} 1111 & \text{if } p \notin w(0), \\ 0000 & \text{if } p \in w(0), \end{cases}$
- $V^{\mathrm{RD}}(w, k, \varphi_0 \wedge \varphi_1) = \min\{V^{\mathrm{RD}}(w, k, \varphi_0), V^{\mathrm{RD}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{RD}}(w, k, \varphi_0 \vee \varphi_1) = \max\{V^{\mathrm{RD}}(w, k, \varphi_0), V^{\mathrm{RD}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{RP}}(w, k, \Diamonddown \varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{j \in \mathbb{N}} V_i^{\mathrm{RP}}(w[j, \infty), k, \varphi)$, and
- $V^{\mathrm{RP}}(w, k, \boxdot \varphi) = b_1 b_2 b_3 b_4$ where
    - $b_1 = \min_{j \in \mathbb{N}} V_1^{\mathrm{RP}}(w[j, \infty), k, \varphi)$, i.e., $b_1 = 1$ iff φ holds always,
    - $b_2 = \max_{j' \in \mathbb{N}} \min_{j \leq j'} V_2^{\mathrm{RP}}(w[j, \infty), k, \varphi)$, i.e., $b_2 = 1$ iff φ holds almost always,
    - $b_3 = \min_{j' \in \mathbb{N}} \max_{j \leq j'} V_3^{\mathrm{RP}}(w[j, \infty), k, \varphi)$, i.e., $b_3 = 1$ iff φ holds infinitely often, and
    - $b_4 = \max_{j \in \mathbb{N}} V_4^{\mathrm{RP}}(w[j, \infty), k, \varphi)$ i.e., $b_4 = 1$ iff φ holds at least once.
- $V^{\mathrm{RP}}(w, k, \Diamonddown_{\mathbf{p}} \varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{0 \leq j \leq k} V_i^{\mathrm{RP}}(w[j, \infty), k, \varphi)$.

It is easy to verify that $V^{\mathrm{RP}}(w, k, \varphi)$ is well-defined, i.e., $V^{\mathrm{RP}}(w, k, \varphi) \in \mathbb{B}_4$ for all w, k, and φ.

Example 1. Consider the formula $\varphi = \boxdot \Diamonddown_{\mathbf{p}} s$, where we interpret occurrences of the atomic proposition s as synchronizations. Then, the different degrees of satisfaction of the formula express the following possibilities, when evaluating it with respect to $k \in \mathbb{N}$: (i) the distance between synchronizations is bounded by k, (ii) from some point onwards, the distance between synchronizations is bounded by k, (iii) there are infinitely many synchronizations, and (iv) there is at least one synchronization. Note that the last two possibilities are independent of k, which is explained by simple logical equivalences, e.g., the third possibility reads actually as follows: there are infinitely many positions such that a synchronization occurs within distance k. However, it is easy to see that is equivalent to the property stated above.

In the next two sections, we solve the model checking problem and the synthesis problem for rPrompt-LTL. To this end, we translate every rPrompt-LTL formula into a sequence of five Prompt-LTL formulas that capture the five degrees of satisfaction and violation by making the semantics of the robust always operator explicit. This is a straightforward generalization of the, in the terms of the introduction, reduction-based approach to robust LTL [26].

Lemma 1. *For every rPrompt-LTL formula φ and every $\beta \in \mathbb{B}_4$, there is a Prompt-LTL formula φ_β of size $\mathcal{O}(|\varphi|)$ such that $V^{\mathrm{RP}}(w, k, \varphi) \succeq \beta$ if and only if $V^{\mathrm{P}}(w, k, \varphi_\beta) = 1$.*

Note that the logic rLTL($\boxdot, \Diamonddown$) is a syntactic fragment of rPrompt-LTL, but Prompt-LTL is not, as we disallow the next, until, and release operator. However, as we present a reduction-based approach, we could easily add them while maintaining the result of Lemma 1. We prefer not to do so for the sake of accessibility and brevity.

3.1 Model Checking

Let us now consider the rPrompt-LTL model checking problem, which asks whether all executions of a given finite transition system satisfy a given specification expressed as an Prompt-LTL formula with truth value at least $\beta \in \mathbb{B}_4$. More formally, we assume the system under consideration to be modeled as a (labeled and initialized) transition system $\mathcal{S} = (S, s_I, E, \lambda)$ over P consisting of a finite set S of states containing the initial state s_I, a directed edge relation $E \subseteq S \times S$, and a state labeling $\lambda \colon S \to 2^P$ that maps each state to the set of atomic propositions that hold true in this state. A path through $\mathcal{S}$ is a sequence $\rho = s_0 s_1 s_2 \cdots$ satisfying $s_0 = s_I$ and $(s_i, s_{i+1}) \in E$ for every $i \in \mathbb{N}$, and $\Pi_{\mathcal{S}}$ denotes the set of all paths through $\mathcal{S}$. Finally, the trace of a path $\rho = s_0 s_1 s_2 \cdots \in \Pi_{\mathcal{S}}$ is the sequence $\lambda(\rho) = \lambda(s_0)\lambda(s_1)\lambda(s_2)\cdots$ of labels induced by ρ.

Problem 1. Let φ be an rPrompt-LTL formula, $\mathcal{S}$ a transition system, and $\beta \in \mathbb{B}_4$. Is there a $k \in \mathbb{N}$ such that $V^{\mathrm{RP}}(\lambda(\rho), k, \varphi) \succeq \beta$ holds true for all paths $\rho \in \Pi_{\mathcal{S}}$?

Our solution relies on Lemma 1 and the fact that Prompt-LTL model checking is in PSPACE [17].

Theorem 1. *rPrompt-LTL model checking is in* PSPACE.

We do not claim PSPACE-hardness, since model checking the fragment of LTL with disjunction, conjunction, always, and eventually operators (and classical semantics) is NP-complete [3]. As this fragment can be embedded into rPrompt-LTL [26], we obtain at least NP-hardness for Problem 1. As we have no next, until, and release operators (by our own volition), we cannot easily claim PSPACE-hardness. In contrast, the solution of the Prompt-LTL model checking problem consists of a reduction to LTL model checking that introduces until operators (see [17]). Hence, we leave the fragment mentioned above, for which NP membership is known. However, adding next, until, and release to rPrompt-LTL yields a PSPACE-hard model checking problem.

3.2 Synthesis

Next, we consider the problem of synthesizing reactive controllers from rPrompt-LTL specifications. In this context, we rely on the classical reduction from reactive synthesis to infinite-duration two-player games over finite graphs. In particular, we show how to construct a finite-state winning strategy for games with rPrompt-LTL winning conditions, which immediately correspond to implementations of reactive controllers. Throughout this section, we assume familiarity with games over finite graphs (see, e.g., [15, Chapter 2]).

We consider rPrompt-LTL games over P, which are triples $\mathcal{G} = (G, \varphi, \beta)$ consisting of a labeled game graph G, an rPrompt-LTL formula φ, and a truth value $\beta \in \mathbb{B}_4$. A labeled game graph $G = (V_0, V_1, E, \lambda)$ consists of a directed graph $(V_0 \cup V_1, E)$, two finite, disjoint sets of vertices V_0 and V_1, and a function $\lambda \colon V_0 \cup V_1 \to 2^P$ mapping each vertex v to the set $\lambda(v)$ of atomic propositions that hold true in v. We denote the set of all vertices by $V = V_0 \cup V_1$ and assume that game graphs do not have terminal vertices, i.e., $\{v\} \times V \cap E \neq \emptyset$ for each $v \in V$.

As in the classical setting, rPrompt-LTL games are played by two players, Player 0 and Player 1, who move a token along the edges of the game graph ad infinitum (if the token is currently placed on a vertex $v \in V_i$, $i \in \{0,1\}$, then Player i decides the next move). The resulting infinite sequence $\rho = v_0 v_1 v_2 \cdots \in V^{\omega}$ of vertices is called a *play* and induces a trace $\lambda(\rho) = \lambda(v_0)\lambda(v_1)\lambda(v_2)\cdots \in (2^P)^{\omega}$. The evaluation of φ on $\lambda(\rho)$ determines the winner of a play: Player 0 wins if $V^{\mathrm{RP}}(\lambda(\rho), \varphi) \succeq \beta$, whereas Player 1 wins if $V^{\mathrm{RP}}(\lambda(\rho), \varphi) \prec \beta$.

Let $i \in \{0,1\}$. A strategy of Player i is a mapping $f \colon V^* V_i \to V$ that prescribes where to move the token depending on the finite play constructed so far. A play $v_0 v_1 v_2 \cdots$ is played according to f if $v_{j+1} = f(v_0 \cdots v_j)$ for every j with $v_j \in V_i$. A strategy f of Player i is winning from a vertex $v \in V$ if all plays that start in v and that are played according to f are winning for Player i. Further, a (winning) strategy is a finite-state strategy if there exists a finite-state machine computing it in the usual sense (see [15, Chapter 2] for details).

We are interested in solving rPrompt-LTL games, i.e., in solving the following problem.

Problem 2. Let $\mathcal{G}$ be an rPrompt-LTL game and v a vertex. Determine whether Player 0 has a winning strategy for $\mathcal{G}$ from v and compute a finite-state winning strategy if so.

Again, our solution to this problem relies on Lemma 1 and the fact that solving Prompt-LTL games is in 2ExpTime [17,31].

Theorem 2. *Solving rPrompt-LTL games is* 2ExpTime-*complete.*

Here we have a matching lower bound, as solving games with LTL conditions without next, until, and release is already 2ExpTime-hard [2].

4   Robust Linear Dynamic Logic

Next, we "robustify" LDL by generalizing the ideas underlying robust LTL to LDL, obtaining the logic rLDL. Again, following the precedent of robust LTL, we equip robust operators with dots to distinguish them from non-robust ones. The formulas of rLDL are given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \langle\cdot r\cdot\rangle\,\varphi \mid [\cdot r\cdot]\,\varphi \qquad\qquad r ::= \phi \mid \varphi? \mid r + r \mid r\,;r \mid r^*,$$

where p ranges over the atomic propositions in P and ϕ over propositional formulas over P. We refer to formulas of the form $\langle\cdot r\cdot\rangle\,\varphi$ and $[\cdot r\cdot]\,\varphi$ as diamond formulas and box formulas, respectively. In both cases, r is the guard of the operator. An atom $\varphi?$ of a regular expression is a test. We use the abbreviations $\mathtt{tt} = p \vee \neg p$ and $\mathtt{ff} = p \wedge \neg p$ for some $p \in P$ and note that both are formulas and guards. We denote the set of subformulas of φ by $\mathrm{cl}(\varphi)$. Guards are not subformulas, but the formulas appearing in the tests are, e.g., we have $\mathrm{cl}(\langle\cdot p?\,;q\cdot\rangle\,p') = \{p, p', \langle\cdot p?\,;q\cdot\rangle\,p'\}$. The size $|\varphi|$ of φ is the sum of $|\mathrm{cl}(\varphi)|$ and the sum of the lengths of the guards appearing in φ (counted with multiplicity and measured in the number of operators).

Before we introduce the semantics of rLDL we first recall the semantics of the robust always operator $\boxdot\varphi$ in robust LTL. To this end, call a position j of a trace φ-satisfying if the suffix starting at position j satisfies φ. Now, the robust semantics are based on the following five cases, where the latter four distinguish various degrees of violating the formula $\square\varphi$: either all positions are φ-satisfying ($\square$), almost all positions are φ-satisfying ($\diamondsuit\square$), infinitely many positions are φ-satisfying ($\square\diamondsuit$), some position is φ-satisfying ($\diamondsuit$), or no position is φ-satisfying.

A similar approach for a formula $[\cdot r\cdot]\,\varphi$ would be to consider the following possibilities, where a position j of a trace w is an r-match if the prefix of w up to and including position $j-1$ is in the language of r: all r-matches are φ-satisfying, almost all r-matches are φ-satisfying, infinitely many r-matches are φ-satisfying, some r-match is φ-satisfying, or no r-match is φ-satisfying. On a trace w with infinitely many r-matches, this is the natural generalization of the robust semantics. A trace, however, may only contain finitely many r-matches, or none at all. In the former case, there are not infinitely many φ-satisfying r-matches, but all r-matches could satisfy φ. Thus, the monotonicity of the cases is violated. We overcome this by interpreting "almost all" as "all" and "infinitely many" as "some" if there are only finitely many r-matches.[4]

Also, the guard r may contain tests, which have to be evaluated to determine whether a position is an r-match. For this, we have to use the appropriate semantics for the robust box operator. For example, if we interpret $[\cdot r\cdot]\,\varphi$ to mean "almost all r-matches satisfy φ", then the robust box operators in tests of r are evaluated with this interpretation as well. This may, however, violate monotonicity (see Example 3), which we therefore hardcode in the semantics.

We now formalize the informal description above and subsequently show that this formalization satisfies all desired properties. To this end, we again define an evaluation function V^{RD} mapping a trace w and a formula φ to a truth value. Also, we again denote the projection of $V^{\mathrm{RD}}(w, \varphi)$ to its i-th bit by $V_i^{\mathrm{RD}}(w, \varphi)$. For atomic propositions and Boolean connectives, the definition is the same as for rPrompt-LTL introduced above (ignoring the bound k) and for negation and implication, the definition is the same as for robust LTL (cf. [26]):

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[4] There is an alternative definition inspired by the semantics of LTL on finite traces: Here, both $\diamondsuit\square\varphi$ and $\square\diamondsuit\varphi$ are equivalent to "φ holds at the last position". This suggests interpreting "almost all r-matches are φ-satisfying" and "infinitely many r-matches are φ-satisfying" as "the last r-match is φ-satisfying" in case there are only finitely many r-matches. Arguably, this definition is less intuitive than the one we propose to pursue.

$$- \ V^{\mathrm{RD}}(w, \neg\varphi) = \begin{cases} 0000 & \text{if } V^{\mathrm{RD}}(w, \varphi) = 1111, \\ 1111 & \text{if } V^{\mathrm{RD}}(w, \varphi) \neq 1111, \end{cases} \text{ and}$$

$$- \ V^{\mathrm{RD}}(w, \varphi_0 \to \varphi_1) = \begin{cases} 1111 & \text{if } V^{\mathrm{RD}}(w, \varphi_0) \preceq V^{\mathrm{RD}}(w, \varphi_1), \\ V^{\mathrm{RD}}(w, \varphi_1) & \text{if } V^{\mathrm{RD}}(w, \varphi_0) \succ V^{\mathrm{RD}}(w, \varphi_1). \end{cases}$$

To define the semantics of the diamond and the box operator, we need to first define the semantics of the guards: The match set $\mathcal{R}_i^{\mathrm{R}}(w, r) \subseteq \mathbb{N}$ for $i \in \{1, 2, 3, 4\}$ contains all positions j of w such that $w[0, j)$ matches r and is defined inductively as follows:

- $\mathcal{R}_i^{\mathrm{R}}(w, \phi) = \{1\}$ if $w(0) \models \phi$ and $\mathcal{R}_i^{\mathrm{R}}(\phi, w) = \emptyset$ otherwise, for propositional ϕ.
- $\mathcal{R}_i^{\mathrm{R}}(w, \varphi?) = \{0\}$ if $V_i^{\mathrm{RD}}(w, \varphi) = 1$ and $\mathcal{R}_i^{\mathrm{R}}(w, \varphi?) = \emptyset$ otherwise.
- $\mathcal{R}_i^{\mathrm{R}}(w, r_0 + r_1) = \mathcal{R}_i^{\mathrm{R}}(w, r_0) \cup \mathcal{R}_i^{\mathrm{R}}(w, r_1)$.
- $\mathcal{R}(w, r_0 \, ; r_1) = \{j_0 + j_1 \mid j_0, j_1 \geq 0 \text{ and } j_0 \in \mathcal{R}(w, r_0) \text{ and } j_1 \in \mathcal{R}(w[j_0, \infty), r_1)\}$, i.e., for j to be in $\mathcal{R}(w, r_0 \, ; r_1)$, it has to be the sum of natural numbers j_0 and j_1 such that w has a prefix of length j_0 that matches r_0 and $w[j_0, \infty)$ has a prefix of length j_1 that matches r_1.
- $\mathcal{R}(w, r^*) = \{0\} \cup \{j_1 + \cdots + j_\ell \mid 0 \geq j_{\ell'} \in \mathcal{R}(w[j_1 + \cdots + j_{\ell'-1}, \infty), r) \text{ for all } \ell' \in \{1, \ldots, \ell\}\}$, where we use $j_1 + \cdots + j_0 = 0$. Thus, for j to be in $\mathcal{R}(w, r^*)$, it has to be expressible as $j = j_1 + \cdots + j_\ell$ with non-negative $j_{\ell'}$ such that the prefix of w of length j_1 matches r, the prefix of length j_2 of $w[j_1, \infty)$ matches r, and in general, the prefix of length $j_{\ell'}$ of $w[j_1 + \cdots + j_{\ell'-1}, \infty)$ matches r, for every $\ell' \in \{1, \ldots, \ell\}$.

Due to tests, membership of j in $\mathcal{R}_i^{\mathrm{R}}(w, r)$ does, in general, not only depend on the prefix $w[0, j)$, but on the complete trace w. Also, the semantics of the propositional atom ϕ differs from the semantics of the test $\phi?$: the former consumes an input letter, while tests do not. Thus, rLDL (as LDL) features both kinds of atoms. We define the intuition given above via

- $V^{\mathrm{RD}}(w, \langle \cdot r \cdot \rangle \varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{j \in \mathcal{R}_i^{\mathrm{R}}(w, r)} V_i^{\mathrm{RD}}(w[j, \infty), \varphi)$ and
- $V^{\mathrm{RD}}(w, [\cdot r \cdot] \varphi) = b_1 b_2 b_3 b_4$ with $b_i = \max\{b'_1, \ldots, b'_i\}$ for every $i \in \{1, 2, 3, 4\}$, where
  - $b'_1 = \min_{j \in \mathcal{R}_1^{\mathrm{R}}(w, r)} V_1^{\mathrm{RD}}(w[j, \infty), \varphi)$,
  - $b'_2 = \begin{cases} \max_{j' \in \mathbb{N}} \min_{j \in \mathcal{R}_2^{\mathrm{R}}(w, r) \cap \{j', j'+1, j'+2, \ldots\}} V_2^{\mathrm{RD}}(w[j, \infty), \varphi) & \text{if } |\mathcal{R}_2^{\mathrm{R}}(w, r)| = \infty, \\ \min_{j \in \mathcal{R}_2^{\mathrm{R}}(w, r)} V_2^{\mathrm{RD}}(w[j, \infty), \varphi) & \text{if } 0 < |\mathcal{R}_2^{\mathrm{R}}(w, r)| < \infty, \\ 1 & \text{if } |\mathcal{R}_2^{\mathrm{R}}(w, r)| = 0, \end{cases}$
  - $b'_3 = \begin{cases} \min_{j' \in \mathbb{N}} \max_{j \in \mathcal{R}_3^{\mathrm{R}}(w, r) \cap \{j', j'+1, j'+2, \ldots\}} V_3^{\mathrm{RD}}(w[j, \infty), \varphi) & \text{if } |\mathcal{R}_3^{\mathrm{R}}(w, r)| = \infty, \\ \max_{j \in \mathcal{R}_3^{\mathrm{R}}(w, r)} V_3^{\mathrm{RD}}(w[j, \infty), \varphi) & \text{if } 0 < |\mathcal{R}_3^{\mathrm{R}}(w, r)| < \infty, \\ 1 & \text{if } |\mathcal{R}_3^{\mathrm{R}}(w, r)| = 0, \end{cases}$
  - $b'_4 = \begin{cases} \max_{j \in \mathcal{R}_4^{\mathrm{R}}(w, r)} V_4^{\mathrm{RD}}(w[j, \infty), \varphi) & \text{if } |\mathcal{R}_4^{\mathrm{R}}(w, r)| > 0, \\ 1 & \text{if } |\mathcal{R}_4^{\mathrm{R}}(w, r)| = 0. \end{cases}$

To give an intuitive description of the semantics, let us first generalize the notion of r-matches and φ-satisfiability. We say that a position j of w is an r-match of degree β if $j \in \mathcal{R}_i^{\mathrm{R}}(w, r)$ for the unique i with $\beta = 0^{i-1} 1^{5-i}$, which requires all tests in r to be evaluated w.r.t. V_i^{RD} (i.e., to some truth value at least β). Similarly, we say that a position j of w is φ-satisfying of degree β if $V^{\mathrm{RD}}(w[j, \infty), \varphi) \succeq \beta$, or if, equivalently, $V_i^{\mathrm{RD}}(w[j, \infty), \varphi) = 1$ for the unique i with $\beta = 0^{i-1} 1^{5-i}$.

Now, consider the b'_i defining the semantics of the robust box operator: We have $b'_1 = 1$ if all r-matches of degree 1111 are φ-satisfying of degree 1111. This is in particular satisfied if there is no such match. Further, if there are infinitely (finitely) many r-matches of degree 0111, then $b'_2 = 1$ if almost all (if all) those matches are φ-satisfying of degree 0111. Dually, if there are infinitely (finitely) many r-matches of degree 0011, then $b'_3 = 1$ if infinitely many (at least one) of those matches are (is) φ-satisfying of degree 0011. Finally, if there is at least one r-match of degree 0001, then $b'_4 = 1$ if at least one of those matches is φ-satisfying of degree 0001. The cases where there is no r-match are irrelevant due to monotonicity, so we hardcode them to 1.

Example 2. Consider the formula $[\cdot r \cdot] q \to [\cdot \mathtt{tt} \, ; r \cdot] p$ with $r = (\mathtt{tt}; \mathtt{tt})^*$, which expresses that the degree of violation of q at *even* positions should at most be the degree of violation of p at *odd* positions. Such a property cannot be expressed in rLTL($\square, \lozenge$), as even $[\cdot r \cdot] q$ is known to be inexpressible in LTL [4].

First, we state that the semantics is well-defined. This is not obvious due to the case distinctions and the use of the matching sets $\mathcal{R}_i^R$ for different i.

Lemma 2. *We have $V^{RD}(w, \varphi) \in \mathbb{B}_4$ for every trace w and every formula φ.*

To conclude, let us give an example witnessing that the maximization over the b_i' in the semantics of the box operator is indeed necessary to obtain monotonicity.

Example 3. Let $\varphi = [\cdot r \cdot]\, \texttt{ff}$ with $r = ([\cdot \texttt{tt}^* \cdot]\, p)?$ and consider the trace $w = \emptyset \{p\}^\omega$. Then, we have $V^{RD}(w, [\cdot \texttt{tt}^* \cdot]\, p) = 0111$ and consequently $\mathcal{R}_1^R(w, r) = \emptyset$ and $\mathcal{R}_2^R(w, r) = \{0\}$. Hence, $\min_{j \in \mathcal{R}_1^R(w, r)} V_1^{RD}(w[j, \infty), \texttt{ff}) = \min \emptyset = 1$, but $\min_{j \in \mathcal{R}_2^R(w, r)} V_2^{RD}(w[j, \infty), \texttt{ff}) = \min\{0\} = 0$. Thus, the bits b_i' inducing $V^{RD}(w, [\cdot r \cdot]\, \texttt{ff})$ are not monotonic, which explains the need to maximize over the b_i' to obtain the semantics of the robust box operator. The traces $(\emptyset \{p\})^\omega$ and $\{p\} \emptyset^\omega$ witness the same result for the other V_i^{RD}.

We prove that rLDL has the exponential compilation property. This allows us to solve the model checking and the synthesis problem using well-known and efficient automata-based algorithms. Furthermore, we are able to show that the complexity of these algorithms is asymptotically the same as the complexity of the algorithms for plain LDL and LTL.

In the terminology introduced in the introduction, we present a direct translation, i.e., we translate rLDL directly into automata. Formally, we follow the approach by Faymonville and Zimmermann [13], who presented a bottom-up translation of parametric LDL, an extension of LDL with prompt temporal operators, into alternating automata of linear size. These are then translated into Büchi automata of exponential size (see [4] for definitions). Here, we do not have to deal with prompt operators, but instead with the consequences of the five-valued semantics.

Theorem 3. *Let φ be an rLDL formula, $n = |\varphi|$, and $\beta \in \mathbb{B}_4$. There is a non-deterministic Büchi automaton $\mathfrak{B}_{\varphi, \beta}$ with $2^{\mathcal{O}(n \log n)}$ states recognizing the language $\{w \in (2^P)^\omega \mid V^{RD}(w, \varphi) \succeq \beta\}$.*

Furthermore, as it is done for the similar construction for PLDL [13], one can show that the automata can indeed be constructed efficiently: the non-deterministic Büchi automaton $\mathfrak{B}_{\varphi, \beta}$ can be constructed on-the-fly in polynomial space.

4.1 Expressiveness

In this section, we compare the expressiveness of rLDL to that of $rLTL(\boxdot, \diamondsuit)$ and LDL. Following Tabuada and Neider [26] we focus on the fragment $rLTL(\boxdot, \diamondsuit)$ without next, until and release operators. While the next and until operator could be added easily, the robust semantics of the release operator is incompatible with our definition of the robust box operator. It turns out as expected, that rLDL generalizes $rLTL(\boxdot, \diamondsuit)$ and LDL and is, in a sense, equivalent to LDL.

Theorem 4. *Both $rLTL(\boxdot, \diamondsuit)$ and LDL can be embedded into rLDL.*

As LDL is a syntactic fragment of LDL, we immediately obtain that LTL can be embedded into rLDL and, thus, rLDL inherits the lower bounds of LTL.

Our next theorem states that LDL and rLDL are of equal expressiveness. The direction from LDL to rLDL was shown in Theorem 4, hence we focus on the other one. Following Tabuada and Neider [26], we construct for every rLDL formula φ and every $i \in \{1, 2, 3, 4\}$ an LDL formula φ_i such that $V_i^{RD}(w, \varphi) = V^D(w, \varphi_i)$ for all traces w. The construction relies on Theorem 3, unlike the analogous result translating robust LTL directly into LTL [26].

Theorem 5. *LDL and rLDL are equally expressive and the translations are effective.*

In general, translating an rLDL formula φ into an equivalent LDL formula φ_i incurs a triply-exponential blow-up, when using the translation described in the proof. On a more positive note, the resulting LDL formula is test-free, i.e., it does not contain tests in its guards. We leave the question of whether there are non-trivial lower bounds on the translation for future work. For the special case of translating $rLTL(\boxdot, \diamondsuit)$ into LTL mentioned above, there is only a linear blowup. This translation was presented by Tabuada and Neider [26], but they only claimed an exponential upper bound. However, closer inspection shows that it is linear if the size of formulas is measured in the number of distinct subformulas, not the length of the formula.

4.2 Model Checking and Synthesis

Theorem 5 immediately provides solutions for typical applications of rLDL, such as model checking and synthesis, by reducing the problem from the domain of rLDL to that of LDL. However, the price to pay for this approach is a triply-exponential blow-up in the size of the resulting LDL formula, which is clearly prohibitive for any real-world application. For this reason, we now develop more efficient model checking and synthesis techniques that are based on our direct translation of rLDL into automata (Theorem 3).

We begin with the rLDL model checking checking problem, which is defined as follows.

Problem 3. Let φ be an rLDL formula, let $\mathcal{S}$ be a transition system, and let $\beta \in \mathbb{B}_4$. Does $V^{\mathrm{RD}}(\lambda(\rho), \varphi) \succeq \beta$ hold true for all paths $\rho \in \Pi_{\mathcal{S}}$?

The exponential compilation property (see Theorem 3) and standard on-the-fly techniques for checking emptiness of exponentially-sized Büchi automata [28] yield a PSPACE upper bound on the complexity of Problem 3. The matching lower bound follows from the subsumption of LDL shown above, as model checking LDL is PSPACE-complete.

Theorem 6. *rLDL model checking is* PSPACE-*complete.*

Similar to model checking, the translation from rLDL formulas to automata provides us with an effective means to synthesize reactive controllers from rLDL specifications, i.e., for the following problem, where an rLDL game now has the form (G, φ, β) and Player 0 wins a play if and only if its trace w satisfies $V^{\mathrm{RD}}(w, \varphi) \geq \beta$.

Problem 4. Let $\mathcal{G}$ be an rLDL game and v a vertex. Determine whether Player 0 has a winning strategy for $\mathcal{G}$ from v and compute a finite-state winning strategy if so.

Theorem 3 provides a straightforward way to solve Problem 4 by reducing it to solving classical parity games (again, see [15, Chapter 2] for an introduction to parity games) while the lower bound follows from the subsumption of LDL.

Theorem 7. *Solving rLDL games is* 2EXPTIME-*complete.*

5 Towards Robust and Prompt Linear Dynamic Logic

In the previous sections, we studied robust LDL, i.e., we combined robustness and increased expressiveness, and robust Prompt-LTL, i.e., we combined robustness and quantitative operators. The third combination of two aspects, i.e., quantitative operators and increased expressiveness, has been studied before [13]. For all three resulting logics, model checking and synthesis have the same complexity as for plain LTL.

Here, we consider the combination of all three extensions, obtaining the logic rPrompt-LDL, robust Prompt-LDL. The syntax is obtained by adding the prompt diamond operator $\langle \cdot r \cdot \rangle_{\mathbf{p}} \varphi$ to LDL, by restricting negations to atomic formulas, and by disallowing implications. Here, r is a guard as in rLDL, which may contain tests, i.e., formulas of rPrompt-LDL. Similarly, the semantics is defined as expected, i.e., it is obtained by extending the semantics of rLDL with a bound k for the prompt diamond operator $\langle \cdot r \cdot \rangle_{\mathbf{p}} \varphi$. Now, its semantics requires the existence of a φ-satisfying r-match within the next k steps. Formal definitions are presented in Appendix F.

Example 4. Consider the formula $[\cdot((\neg t)^*; t; (\neg t)^*; t)^* \cdot] \langle \cdot \mathtt{tt}^* \cdot \rangle_{\mathbf{p}} s$ and interpret t as the tick of a clock and s as a synchronization. Then, the formula intuitively expresses that every other tick of the clock is followed after a bounded number of steps (not ticks!) by a synchronization.

More formally, the different degrees of satisfaction of φ express the following possibilities, with respect to a given bound k: (i) every even clock tick is followed by a synchronization within k steps; (ii) almost every even clock tick is followed by a synchronization within k steps; (iii) infinitely many even clock ticks are followed by a synchronization within k steps; (iv) there is at least one even clock tick that is followed by a synchronization within k steps.

This property can neither be expressed in (robust) LDL nor in (robust) Prompt-LTL. Also note that unlike for the similar formula from Example 1, the last two possibilities are not trivial, as we now only consider positions with an even clock tick and not all positions.

In the previous sections, we have seen two approaches to translating robust logics into Büchi automata, the direct and the reduction-based one. Both are extensions of translations originally introduced by Tabuada and Neider for robust LTL. The former one translates a formula of a robust logic directly into an equivalent Büchi automaton while the latter one first translates a formula of a robust logic into an equivalent classical (non-robust) logic, for which a translation into equivalent Büchi automata is already known. For robust LTL, both approaches are applicable [26] and yield Büchi automata of exponential size. Here, we, out of necessity, apply both approaches: for robust LDL, we present a direct translation while we present a reduction-based approach for robust Prompt-LTL. Let us quickly elaborate the reasons for this.

First, consider the reduction-based approach for robust LTL, which translates a formula φ of robust LTL and a truth value $\beta \succ 0000$ into an LTL formula φ_β that captures φ with respect to β. To this end, the formula φ_β implements the intuitive meaning of the robust semantics for the always operator, e.g., we have $(\boxdot p)_{1111} = \Box p$, $(\boxdot p)_{0111} = \Diamond \Box p$, $(\boxdot p)_{0011} = \Box \Diamond p$, and $(\boxdot p)_{0001} = \Diamond p$.

Trying to apply this approach to the rLDL formula $\varphi = [\cdot r \cdot] p$, say for $\beta = 0111$, would imply to use a formula of the form $\langle \cdot r_0 \cdot \rangle [\cdot r_1 \cdot] p$ where r_0 and r_1 are obtained by "splitting" up r. Formally, consider a trace w and an arbitrary r-match j. Then, to capture the robust semantics of φ with respect to β, for every $j' \leq j$, j' has to be an r_0-match in w and $j - j'$ has to be an r_1-match in $w[j', \infty)$. Further, to obtain a translation of optimal complexity, r_0 and r_1 have to be of polynomial size in $|r|$. It is an open problem whether such a splitting is always possible, in particular in the presence of tests in r and guards with only finitely many r-matches.

Secondly, recall the direct approach to robust LTL, which translates a formula φ of rLTL($\boxdot, \Diamond$) into a Büchi automaton that captures φ with respect to all $\beta \in \mathbb{B}_4$ (by considering five initial states, one for each β). Trying to apply this approach to robust Prompt-LTL requires using a more general automaton model that is able to capture the quantitative nature of the prompt diamond operator while still yielding a model checking and a synthesis algorithm with the desired complexity. To the best of our knowledge, no such translation from Prompt-LTL to automata has been presented in the literature, which would be a special case of our construction here.

Thus, according to the state-of-the-art, the direct approach is the only viable one for robust extensions of LDL while the reduction-based approach is the only viable one for robust extensions of Prompt-LTL. This leaves us with no viable approach for rPrompt-LDL.

Nevertheless, in Appendix F, we present a fragment of rPrompt-LDL and a reduction-based translation to Prompt-LDL for it. The fragment is obtained by disallowing tests in guards and requiring them to always have infinitely many matches. For such formulas, one can translate the guard into a deterministic finite automaton (without tests) and then use this automaton to "split" r. However, this involves multiple exponential blowups and hence does not prove that the fragment has the exponential compilation property. Nonetheless, this translation shows that both model checking and synthesis are decidable for this fragment. The decidability of these problems for full rPrompt-LDL is left for further research and seemingly requires new approaches.

6 Conclusion

We addressed the problems of verification and synthesis with robust, expressive, and quantitative linear temporal specifications. Inspired by robust LTL, we have first developed robust extensions of the logics LDL and Prompt-LTL, named rLDL and rPrompt-LTL, respectively. Then, we combined rLDL and rPrompt-LTL into a third logic, named rPrompt-LDL, which has the expressiveness of ω-regular languages and allows robust reasoning about timing bounds.

For rLDL and rPrompt-LTL, we have shown how to solve the model checking and synthesis problem relying on the exponential compilation property. Hence, all these problems are not harder than those for plain LTL. The situation for the combination of all three basic logics, i.e., for rPrompt-LDL, is less encouraging. In Appendix F, we show the problems to be decidable for an important fragment, but due to a blowup of the formulas during the reduction, we (most likely) do not obtain optimal algorithms. Decidability for the full logic remains open.

In future work, we aim to determine the exact complexity of the model checking and synthesis problem for (full) rPrompt-LDL. One promising approach is to generalize the translation of rLDL into alternating parity automata. However, this requires a suitable quantitative alternating automata model with strong closure properties that can be transformed into equivalent non-deterministic and deterministic automata.

Another promising direction for further research is to study the semantics for the robust box operator proposed in Footnote 4 on Page 7. In particular, it is open whether the translation into alternating automata can be generalized to this setting without a blowup. Also, we leave open whether full robust LTL, i.e., with until and release, can be embedded into rLDL. As is, the robust semantics of the release operator (see [26]) is not compatible with our robust semantics for rLDL. In future work, we plan to study generalizations of full robust LTL.

References

1. Alur, R., Etessami, K., Torre, S.L., Peled, D.: Parametric temporal logic for "model measuring". ACM Trans. Comput. Log. 2(3), 388–407 (2001)
2. Alur, R., La Torre, S., Madhusudan, P.: Playing games with boxes and diamonds. In: Amadio, R.M., Lugiez, D. (eds.) CONCUR 2003. LNCS, vol. 2761, pp. 127–141. Springer (2003)
3. Alur, R., Torre, S.L.: Deterministic generators and games for LTL fragments. ACM Trans. Comput. Log. 5(1), 1–25 (2004)
4. Baier, C., Katoen, J.P.: Principles of Model Checking. The MIT Press (2008)
5. Bloem, R., Chatterjee, K., Greimel, K., Henzinger, T.A., Jobstmann, B.: Robustness in the presence of liveness. In: CAV 2010. LNCS, vol. 6174, pp. 410–424. Springer (2010)
6. Calude, C.S., Jain, S., Khoussainov, B., Li, W., Stephan, F.: Deciding parity games in quasipolynomial time. In: STOC 2017. pp. 252–263. ACM (2017)
7. Dallal, E., Neider, D., Tabuada, P.: Synthesis of safety controllers robust to unmodeled intermittent disturbances. In: CDC 2016. pp. 7425–7430 (2016)
8. De Giacomo, G., Vardi, M.Y.: Linear temporal logic and linear dynamic logic on finite traces. In: Rossi, F. (ed.) IJCAI. IJCAI/AAAI (2013)
9. Donzé, A., Maler, O.: Robust satisfaction of temporal logic over real-valued signals. In: Chatterjee, K., Henzinger, T.A. (eds.) FORMATS 2010. LNCS, vol. 6246, pp. 92–106. Springer (2010)
10. Doyen, L., Henzinger, T.A., Legay, A., Nickovic, D.: Robustness of sequential circuits. In: Gomes, L., Khomenko, V., Fernandes, J.M. (eds.) ACSD 2010. pp. 77–84. IEEE Computer Society (2010)
11. Eisner, C., Fisman, D.: A Practical Introduction to PSL. Integrated Circuits and Systems, Springer (2006)
12. Fainekos, G.E., Pappas, G.J.: Robustness of temporal logic specifications for continuous-time signals. Theor. Comput. Sci. 410(42), 4262–4291 (2009)
13. Faymonville, P., Zimmermann, M.: Parametric linear dynamic logic. Inf. Comput. 253, 237–256 (2017)
14. Fix, L.: Fifteen years of formal property verification in intel. In: Grumberg, O., Veith, H. (eds.) 25 Years of Model Checking - History, Achievements, Perspectives. LNCS, vol. 5000, pp. 139–144. Springer (2008)
15. Grädel, E., Thomas, W., Wilke, T. (eds.): Automata, Logics, and Infinite Games: A Guide to Current Research, LNCS, vol. 2500. Springer (2002)
16. Koymans, R.: Specifying real-time properties with metric temporal logic. Real-Time Systems 2, 255–299 (1990)
17. Kupferman, O., Piterman, N., Vardi, M.Y.: From liveness to promptness. Formal Methods in System Design 34(2), 83–103 (2009)
18. Leucker, M., Sánchez, C.: Regular linear temporal logic. In: Jones, C.B., Liu, Z., Woodcock, J. (eds.) ICTAC 2007. LNCS, vol. 4711, pp. 291–305. Springer (2007)
19. Majumdar, R., Saha, I.: Symbolic robustness analysis. In: Baker, T.P. (ed.) RTSS 2009. pp. 355–363. IEEE Computer Society (2009)
20. Neider, D., Weinert, A., Zimmermann, M.: Synthesizing optimally resilient controllers. In: Ghica, D.R., Jung, A. (eds.) CSL 2018. LIPIcs, vol. 119, pp. 34:1–34:17. Schloss Dagstuhl - LZI (2018)
21. Pnueli, A.: The temporal logic of programs. In: FOCS 1977. pp. 46–57. IEEE (Oct 1977)
22. Pnueli, A., Rosner, R.: On the synthesis of an asynchronous reactive module. In: Ausiello, G., Dezani-Ciancaglini, M., Rocca, S.R.D. (eds.) ICALP 1989. LNCS, vol. 372, pp. 652–671. Springer (1989)
23. Sistla, A.P., Clarke, E.M.: The complexity of propositional linear temporal logics. J. ACM 32(3), 733–749 (1985)
24. Sistla, A.P., Vardi, M.Y., Wolper, P.: The complementation problem for Büchi automata with applications to temporal logic (extended abstract). In: Brauer, W. (ed.) ICALP 1985. LNCS, vol. 194, pp. 465–474. Springer (1985)
25. Tabuada, P., Caliskan, S.Y., Rungger, M., Majumdar, R.: Towards robustness for cyber-physical systems. IEEE Trans. Automat. Contr. 59(12), 3151–3163 (2014)
26. Tabuada, P., Neider, D.: Robust linear temporal logic. In: Talbot, J., Regnier, L. (eds.) CSL 2016. LIPIcs, vol. 62, pp. 10:1–10:21. Schloss Dagstuhl - LZI (2016)
27. Vardi, M.Y.: The rise and fall of LTL. In: D'Agostino, G., Torre, S.L. (eds.) GandALF 2011. EPTCS, vol. 54 (2011)

28. Vardi, M.Y., Wolper, P.: Reasoning about infinite computations. Inf. Comput. 115(1), 1–37 (1994)
29. Weinert, A., Zimmermann, M.: Visibly linear dynamic logic. In: Lal, A., Akshay, S., Saurabh, S., Sen, S. (eds.) FSTTCS 2016. LIPIcs, vol. 65, pp. 28:1–28:14. Schloss Dagstuhl - LZI (2016)
30. Wolper, P.: Temporal logic can be more expressive. Information and Control 56(1/2), 72–99 (1983)
31. Zimmermann, M.: Optimal bounds in parametric LTL games. Theor. Comput. Sci. 493, 30–45 (2013)
32. Zimmermann, M.: Parameterized linear temporal logics meet costs: still not costlier than LTL. Acta Inf. 55(2), 129–152 (2018)

A A Zoo of Temporal Logics

In this section, we introduce the logics we generalize in this work. See Table 1 for an overview. References for the results mentioned in the table are given in the following subsections introducing the logics.

Logic	Operators	Complexity	
		Model Checking	Synthesis
$\text{rLTL}(\boxdot, \Diamonddot)$	$\neg, \wedge, \vee, \rightarrow, \boxdot, \Diamonddot$	NP-hard/in PSPACE	2ExpTime-compl.
LDL	$\neg, \wedge, \vee, \rightarrow, \langle r \rangle, [r]$	PSPACE-compl.	2ExpTime-compl.
Prompt-LTL	$\wedge, \vee, \bigcirc, \mathbf{U}, \mathbf{R}, \Diamond_{\mathbf{P}}$	PSPACE-compl.	2ExpTime-compl.
Prompt-LDL	$\wedge, \vee, \langle r \rangle, [r], \langle r \rangle_{\mathbf{P}}$	PSPACE-compl.	2ExpTime-compl.

Table 1. The logics our work is based on.

A.1 Robust Linear Temporal Logic

When Tabuada and Neider introduced robust LTL [26], they first considered the fragment $\text{rLTL}(\boxdot, \Diamonddot)$ without the next, until, and release operators, which already captures the most interesting problems arising from adding robustness. Then, they added the missing operators and studied the full logic. Here, we follow their approach and only consider generalizations of the fragment $\text{rLTL}(\boxdot, \Diamonddot)$ that only contains the temporal operators $\boxdot$ and $\Diamonddot$.

Formally, the formulas of $\text{rLTL}(\boxdot, \Diamonddot)$ are given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Diamonddot \varphi \mid \boxdot \varphi,$$

where p ranges over the atomic propositions in P. Following the convention introduced in the original paper, we denote the temporal operators of robust logics with dots as to make the distinction to the original logics more visible. Moreover, note that the syntax of $\text{rLTL}(\boxdot, \Diamonddot)$ explicitly contains implication and conjunction; due to the many-valued semantics of $\text{rLTL}(\boxdot, \Diamonddot)$ introduced below, these two operators cannot be recovered from disjunction and negation. We define the size $|\varphi|$ of a formula as the number of distinct subformulas of φ.

Informally, the five-valued robust semantics is based on the following four possibilities a formula $\boxdot \varphi$ can be violated: either φ is violated finitely often (represented by 0111), infinitely often (represented by 0011), almost always (represented by 0001), or always (represented by 0000). Thus, these truth values capture different degrees of "false" while the fifth truth value 1111 captures satisfaction. The semantics of the Boolean operators are lifted to the five-valued setting. For the motivation behind their definition, we refer to the original work by Tabuada and Neider [26].

Formally, the semantics of $\text{rLTL}(\boxdot, \Diamonddot)$ is given by an evaluation function V^{R} mapping a trace w and a formula φ to a truth value from $\mathbb{B}_4$ that is defined as follows:

- $V^{\text{R}}(w, p) = \begin{cases} 1111 & \text{if } p \in w(0), \\ 0000 & \text{if } p \notin w(0), \end{cases}$

- $V^{\text{R}}(w, \neg\varphi) = \begin{cases} 1111 & \text{if } V^{\text{R}}(w, \varphi) \neq 1111, \\ 0000 & \text{if } V^{\text{R}}(w, \varphi) = 1111, \end{cases}$

- $V^{\text{R}}(w, \varphi_0 \wedge \varphi_1) = \min\{V^{\text{R}}(w, \varphi_0), V^{\text{R}}(w, \varphi_1)\}$,

- $V^{\text{R}}(w, \varphi_0 \vee \varphi_1) = \max\{V^{\text{R}}(w, \varphi_0), V^{\text{R}}(w, \varphi_1)\}$,

- $V^{\text{R}}(w, \varphi_0 \rightarrow \varphi_1) = \begin{cases} 1111 & \text{if } V^{\text{R}}(w, \varphi_0) \preceq V^{\text{R}}(w, \varphi_1), \\ V^{\text{R}}(w, \varphi_1) & \text{if } V^{\text{R}}(w, \varphi_0) \succ V^{\text{R}}(w, \varphi_1), \end{cases}$

- $V^{\text{R}}(w, \Diamonddot \varphi) = b_1 b_2 b_3 b_4$ with $b_i = \max_{j \geq 0} V_i^{\text{R}}(w[j, \infty), \varphi)$, and

- $V^{\text{R}}(w, \boxdot \varphi) = b_1 b_2 b_3 b_4$ with
  - $b_1 = \min_{j \geq 0} V_1^{\text{R}}(w[j, \infty), \varphi)$,
  - $b_2 = \max_{j' \in \mathbb{N}} \min_{j \geq j'} V_2^{\text{R}}(w[j, \infty), \varphi)$,

- $b_3 = \min_{j' \in \mathbb{N}} \max_{j \geq j'} V_3^{\mathrm{R}}(w[j, \infty), \varphi)$, and
- $b_4 = \max_{j \geq 0} V_4^{\mathrm{R}}(w[j, \infty), \varphi)$.

Here, $V_i^{\mathrm{R}}(w, \varphi)$ denotes the projection of $V^{\mathrm{R}}(w, \varphi)$ to its i-th component, i.e., we have

$$V^{\mathrm{R}}(w, \varphi) = V_1^{\mathrm{R}}(w, \varphi) V_2^{\mathrm{R}}(w, \varphi) V_3^{\mathrm{R}}(w, \varphi) V_4^{\mathrm{R}}(w, \varphi).$$

The first bit of the semantics captures the classical semantics of LTL, i.e., we have $V_1^{\mathrm{R}}(w, \varphi) = 1$ if and only if w satisfies φ classically. Intuitively, the next three bits are obtained by weakening the semantics of the subformulas of the form $\boxdot \varphi$: instead of (classically) requiring every position to satisfy φ, the second bit is one if almost all positions satisfy φ (i.e., $\Diamond \Box \varphi$ holds), the third bit is one if infinitely many positions satisfy φ (i.e., $\Box \Diamond \varphi$ holds), and the fourth bit is one if at least one position satisfies φ (i.e., $\Diamond \varphi$ holds). Note that negation and implication are also non-classical and break this intuition for subformulas of the form $\boxdot \varphi$ in the scope of a negation or implication. For a full motivation and explanation of the semantics, we refer to the original work introducing rLTL($\boxdot, \Diamonddot$) [26].

Verification problems with rLTL($\boxdot, \Diamonddot$) specifications have a threshold $\beta \in \mathbb{B}_4$ as an additional input and ask every trace to evaluate to at least β. Tabuada and Neider showed that the model checking problem with rLTL($\boxdot, \Diamonddot$) specifications can be solved in polynomial space[5] and that infinite games with rLTL($\boxdot, \Diamonddot$) specifications can be solved in doubly-exponential time [26]. The lower bounds presented in Table 1 are derived from the special case of LTL($\Box, \Diamond$) [2,3], which is a fragment of rLTL($\boxdot, \Diamonddot$).

A.2 Linear Dynamic Logic

The formulas of LDL are given by the grammar

$$\varphi ::= p \mid \neg \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \langle r \rangle \varphi \mid [r] \varphi$$
$$r ::= \phi \mid \varphi? \mid r + r \mid r \,;\, r \mid r^*$$

where p ranges over the atomic propositions in P and where ϕ ranges over arbitrary propositional formulas over P. The regular expressions have two types of atoms: propositional formulas ϕ over the atomic propositions and tests $\varphi?$, where φ is again an LDL formula. As we will see later, the semantics of these two kinds of atoms differ significantly. We refer to formulas of the form $\langle r \rangle \varphi$ and $[r] \varphi$ as diamond formulas and box formulas, respectively. In both cases, we call r the guard of the operator.

We denote the set of subformulas of φ by $\mathrm{cl}(\varphi)$. Guards are not subformulas, but the formulas appearing in the tests are, e.g., we have $\mathrm{cl}(\langle p? \,;\, q \rangle p') = \{p, p', \langle p? \,;\, q \rangle p'\}$. The size $|\varphi|$ of φ is the sum of $|\mathrm{cl}(\varphi)|$ and the sum of the lengths of the guards appearing in φ (counted with multiplicity and measured in the number of operators).

Analogously to the definition for rLTL($\boxdot, \Diamonddot$), and slightly non-standard, we define the semantics of LDL by specifying an evaluation function V^{D} mapping a trace w and a formula φ to a truth value from $\mathbb{B}$ denoting whether w satisfies φ or not. Also, our presentation of the semantics here is slightly cumbersome, in particular the definition for the implication, again to align with the definition for rLTL($\boxdot, \Diamonddot$). Nevertheless, our definition below is equivalent to the classical semantics of LDL (cf. [8,13]) via a satisfaction relation $\models$ in the following sense: we have $V^{\mathrm{D}}(w, \varphi) = 1$ if and only if $w \models \varphi$.

- $V^{\mathrm{D}}(w, p) = \begin{cases} 1 & \text{if } p \in w(0), \\ 0 & \text{if } p \notin w(0), \end{cases}$
- $V^{\mathrm{D}}(w, \neg \varphi) = \begin{cases} 1 & \text{if } V^{\mathrm{D}}(w, \varphi) = 0, \\ 0 & \text{if } V^{\mathrm{D}}(w, \varphi) = 1, \end{cases}$
- $V^{\mathrm{D}}(w, \varphi_0 \wedge \varphi_1) = \min\{V^{\mathrm{D}}(w, \varphi_0), V^{\mathrm{D}}(w, \varphi_1)\}$,
- $V^{\mathrm{D}}(w, \varphi_0 \vee \varphi_1) = \max\{V^{\mathrm{D}}(w, \varphi_0), V^{\mathrm{D}}(w, \varphi_1)\}$,
- $V^{\mathrm{D}}(w, \varphi_0 \rightarrow \varphi_1) = \begin{cases} 1 & \text{if } V^{\mathrm{D}}(w, \varphi_0) \leq V^{\mathrm{D}}(w, \varphi_1), \\ V^{\mathrm{D}}(w, \varphi_1) & \text{if } V^{\mathrm{D}}(w, \varphi_0) > V^{\mathrm{D}}(w, \varphi_1), \end{cases}$
- $V^{\mathrm{D}}(w, \langle r \rangle \varphi) = \max_{j \in \mathcal{R}(w, r)} V^{\mathrm{D}}(w[j, \infty), \varphi)$, and

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[5] Tabuada and Neider only showed that their algorithm runs in exponential time, but using standard on-the-fly techniques [28] it can also be implemented in polynomial space.

- $V^{\mathrm{D}}(w, [r]\,\varphi) = \min_{j \in \mathcal{R}(w,r)} V^{\mathrm{D}}(w[j,\infty), \varphi)$.

Here, the match set $\mathcal{R}(w,r) \subseteq \mathbb{N}$ contains all positions j such that $w[0,j)$ matches r. Recall that $w[0,j)$ denotes the prefix of w of length j, i.e., $w[0,j) = w(0)\cdots w(j-1)$. In particular, $w[0,0)$ is empty and $w[0,\infty)$ is w. Now, $\mathcal{R}(w,r)$ is defined inductively as follows:

- $\mathcal{R}(w,\phi) = \{1\}$ if $w(0) \models \phi$ and $\mathcal{R}(w,\phi) = \emptyset$ otherwise, for propositional ϕ.
- $\mathcal{R}(w,\varphi?) = \{0\}$ if $V^{\mathrm{D}}(w,\varphi) = 1$ and $\mathcal{R}(w,\varphi?) = \emptyset$ otherwise.
- $\mathcal{R}(w, r_0 + r_1) = \mathcal{R}(w, r_0) \cup \mathcal{R}(w, r_1)$.
- $\mathcal{R}(w, r_0\,;r_1) = \{j_0 + j_1 \mid j_0, j_1 \geq 0 \text{ and } j_0 \in \mathcal{R}(w, r_0) \text{ and } j_1 \in \mathcal{R}(w[j_0,\infty), r_1)\}$. Thus, for j to be in $\mathcal{R}(w, r_0\,;r_1)$, it has to be the sum of natural numbers j_0 and j_1 such that w has a prefix of length j_0 that matches r_0 and $w[j_0,\infty)$ has a prefix of length j_1 that matches r_1.
- $\mathcal{R}(w, r^*) = \{0\} \cup \{j_1 + \cdots + j_\ell \mid 0 \leq j_{\ell'} \in \mathcal{R}(w[j_1 + \cdots + j_{\ell'-1}, \infty), r) \text{ for all } \ell' \in \{1, \ldots, \ell\}\}$, where we use $j_1 + \cdots + j_0 = 0$. Thus, for j to be in $\mathcal{R}(w, r^*)$, it has to be expressible as $j = j_1 + \cdots + j_\ell$ with non-negative $j_{\ell'}$ such that the prefix of w of length j_1 matches r, the prefix of length j_2 of $w[j_1,\infty)$ matches r, and in general, the prefix of length $j_{\ell'}$ of $w[j_1 + \cdots + j_{\ell'-1}, \infty)$ matches r, for every $\ell' \in \{1, \ldots, \ell\}$.

Due to tests, membership of some j in $\mathcal{R}(w,r)$ does, in general, not only depend on the prefix $w[0,j)$, but on the complete trace w. Also, the semantics of the propositional atom ϕ differ from the semantics of the test $\phi?$: the former consumes an input letter, while tests do not. Hence, the guards of LDL feature both kinds of atoms.

Fix some trace w, a formula φ, and a guard r. We say that a position j of w is an r-match if $j \in \mathcal{R}(w,r)$. Further, j is a φ-satisfying position of w if $V^{\mathrm{D}}(w[j,\infty), \varphi) = 1$. Thus, the formula $\langle r \rangle\,\varphi$ requires some φ-satisfying r-match to exist. Dually, $[r]\,\varphi$ requires every r-match of w to be φ-satisfying (in particular, this is the case if there is no r-match in w). Thus, the diamond operator generalizes the eventually operator and the box operator generalizes the always operator, which are the respective special cases for a trivial guard that matches every position, e.g., $\mathtt{tt}^*$. Similarly, the next-, until-, and release operator of LTL can be expressed in LDL (the latter two use tests in the guards). Thus, LTL is a fragment of LDL. Furthermore, it is known that LDL captures the ω-regular languages [27].

Model checking against LDL specifications is PSpace-complete and solving LDL games is 2ExpTime-complete [13,27].

A.3 Prompt Linear Temporal Logic

The formulas of Prompt-LTL are given by the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bigcirc \varphi \mid \varphi \,\mathbf{U}\, \varphi \mid \varphi \,\mathbf{R}\, \varphi \mid \Diamond_{\mathbf{p}}\,\varphi$$

where p ranges over the atomic propositions in P. The size $|\varphi|$ of a formula φ is defined as the number of its distinct subformulas.

In Prompt-LTL, formulas are in negation normal form and implication is disallowed. Both requirements are necessary to preserve monotonicity of the prompt-eventually $\Diamond_{\mathbf{p}}$ with respect to the parameter k bounding it (Alur et al. [1] provide a detailed discussion). Again, let $\mathtt{tt} = p \vee \neg p$ and $\mathtt{ff} = p \wedge \neg p$ for some atomic proposition p, which we use to define the shorthands $\Diamond \varphi = \mathtt{tt} \,\mathbf{U}\, \varphi$ and $\square \varphi = \mathtt{ff} \,\mathbf{R}\, \varphi$.

Again, we define the semantics by an evaluation function V^{P} mapping a trace w, a bound $k \in \mathbb{N}$ for the prompt operators, and a formula φ to a truth value in $\mathbb{B}$ (which is again equivalent to the standard definition):

- $V^{\mathrm{P}}(w, k, p) = \begin{cases} 1 & \text{if } p \in w(0), \\ 0 & \text{if } p \notin w(0), \end{cases}$
- $V^{\mathrm{P}}(w, k, \neg p) = \begin{cases} 1 & \text{if } p \notin w(0), \\ 0 & \text{if } p \in w(0), \end{cases}$
- $V^{\mathrm{P}}(w, k, \varphi_0 \wedge \varphi_1) = \min\{V^{\mathrm{P}}(w, k, \varphi_0), V^{\mathrm{P}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{P}}(w, k, \varphi_0 \vee \varphi_1) = \max\{V^{\mathrm{P}}(w, k, \varphi_0), V^{\mathrm{P}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{P}}(w, k, \bigcirc \varphi) = V^{\mathrm{P}}(w[1,\infty), k, \varphi)$,

- $V^{\mathrm{P}}(w, k, \varphi_0 \, \mathbf{U} \, \varphi_1) = \max_{j \in \mathbb{N}} \min\{V^{\mathrm{P}}(w[j, \infty), k, \varphi_1), \min_{0 \le j' < j} V^{\mathrm{P}}(w[j', \infty)), k, \varphi_0\}$,
- $V^{\mathrm{P}}(w, k, \varphi_0 \, \mathbf{R} \, \varphi_1) = \min_{j \in \mathbb{N}} \max\{V^{\mathrm{P}}(w[j, \infty), k, \varphi_1), \max_{0 \le j' \le j} V^{\mathrm{P}}(w[j', \infty)), k, \varphi_0\}$,
- $V^{\mathrm{P}}(w, k, \diamondsuit_{\mathbf{p}} \, \varphi) = \max_{0 \le j \le k} V^{\mathrm{P}}(w[j, \infty), k, \varphi)$.

In verification problems for Prompt-LTL, the bound k on the prompt-eventuallies is existentially quantified. Kupferman et al. proved that Prompt-LTL model checking is PSPACE-complete and that solving games with Prompt-LTL winning conditions is 2EXPTIME-complete [17].[6]

A.4 Prompt Linear Dynamic Logic

In our proofs, we also use Prompt-LDL, which can be seen as a combination of LDL and Prompt-LTL. This logic has been studied by Faymonville and Zimmermann as a fragment of Parametric LTL [1] (although it has never been explicitly named).

The formulas of Prompt-LDL are given by the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle r \rangle \, \varphi \mid [r] \, \varphi \mid \langle r \rangle_{\mathbf{p}} \, \varphi$$
$$r ::= \phi \mid \varphi? \mid r + r \mid r \,; r \mid r^*$$

where p again ranges over the atomic propositions in P and ϕ ranges over propositional formulas over P. As in Prompt-LTL, we have to disallow arbitrary negations and implications. The size of a formula is defined as for LDL.

Furthermore, the semantics of Prompt-LDL are obtained by combining the one of LDL and the one of Prompt-LTL: Again, we define an evaluation function V^{PD} mapping a trace w, a bound k, and a formula φ to a truth value.

- $V^{\mathrm{PD}}(w, k, p) = \begin{cases} 1 & \text{if } p \in w(0), \\ 0 & \text{if } p \notin w(0), \end{cases}$
- $V^{\mathrm{PD}}(w, k, \neg p) = \begin{cases} 1 & \text{if } p \notin w(0), \\ 0 & \text{if } p \in w(0), \end{cases}$
- $V^{\mathrm{PD}}(w, k, \varphi_0 \wedge \varphi_1) = \min\{V^{\mathrm{PD}}(w, k, \varphi_0), V^{\mathrm{PD}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{PD}}(w, k, \varphi_0 \vee \varphi_1) = \max\{V^{\mathrm{PD}}(w, k, \varphi_0), V^{\mathrm{PD}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{PD}}(w, k, \langle r \rangle \, \varphi) = \max_{j \in \mathcal{R}(w, k, r)} V^{\mathrm{PD}}(w[j, \infty), k, \varphi)$,
- $V^{\mathrm{PD}}(w, k, [r] \, \varphi) = \min_{j \in \mathcal{R}(w, k, r)} V^{\mathrm{PD}}(w[j, \infty), k, \varphi)$, and
- $V^{\mathrm{PD}}(w, k, \langle r \rangle_{\mathbf{p}} \, \varphi) = \max_{j \in \mathcal{R}(w, k, r) \cap \{0, \ldots, k\}} V^{\mathrm{PD}}(w[j, \infty), k, \varphi)$.

Here, $\mathcal{R}(w, k, r)$ is defined as $\mathcal{R}(w, r)$, but propagates the bound k to evaluate tests. Hence, we define $\mathcal{R}(w, k, \varphi?) = \{0\}$ if $V^{\mathrm{D}}(w, k, \varphi) = 1$ and $\mathcal{R}(w, k, \varphi?) = \emptyset$ otherwise. All other cases are defined as for LDL, but propagate the bound k.

Prompt-LDL as defined here is a syntactic fragment of Parametric LDL [13] and subsumes LTL. Hence, its model checking problem is PSPACE-complete and the synthesis problem is 2EXPTIME-complete. Here, the bound k is again uniformly existentially quantified in verification problems.

B Proofs from Section 3

Proof (Proof of Lemma 1). If $\beta = 0000$, then we can pick $\varphi_\beta = \mathtt{tt}$, independently of φ. Otherwise, we obtain the result by induction over the construction of φ implementing the intuition behind the robust semantics:

- $p_\beta = p$ and $(\neg p)_\beta = \neg p$ for all atomic propositions $p \in P$ and all $\beta \succ 0000$.
- $(\varphi_0 \wedge \varphi_1)_\beta = (\varphi_0)_\beta \wedge (\varphi_1)_\beta$ for all $\beta \succ 0000$.
- $(\varphi_0 \vee \varphi_1)_\beta = (\varphi_0)_\beta \vee (\varphi_1)_\beta$ for all $\beta \succ 0000$.
- $(\diamondsuit \varphi)_\beta = \diamondsuit (\varphi_\beta)$ for all $\beta \succ 0000$.
- $(\boxdot \varphi)_{1111} = \Box(\varphi_{1111})$.

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[6] Instead of games, they actually considered the related framework of realizability, an abstract type of game without underlying graph. However, realizability and graph-based games are interreducible (also, see [31]).

- $(\boxdot \varphi)_{0111} = \diamondsuit\square(\varphi_{0111})$.
- $(\boxdot \varphi)_{0011} = \square\diamondsuit(\varphi_{0011})$.
- $(\boxdot \varphi)_{0001} = \diamondsuit(\varphi_{0001})$.
- $(\diamondsuit_{\mathbf{P}} \varphi)_{\beta} = \diamondsuit_{\mathbf{P}}(\varphi_{\beta})$ for all $\beta \succ 0000$.

A straightforward induction shows that the resulting formula has the desired properties.

Proof (Proof of Theorem 1). By Lemma 1, there exists a $k \in \mathbb{N}$ such that $V^{\mathrm{RP}}(\lambda(\rho), k, \varphi) \succeq \beta$ holds true for all paths $\rho \in \Pi_{\mathcal{S}}$ if and only if there exists a $k \in \mathbb{N}$ such that $V^{\mathrm{P}}(\lambda(\rho), k, \varphi_{\beta}) = 1$ for all paths $\rho \in \Pi_{\mathcal{S}}$. The latter is an instance of the Prompt-LTL model checking problem, which is in PSPACE [17].

Proof (Proof of Theorem 2). The lower bound follows from the special case of LTL($\square, \diamondsuit$) [2], which is a fragment of rPrompt-LTL. On the other hand, the upper bound is again proven by a reduction to Prompt-LTL: Player 0 having a winning strategy for (G, φ, β) from v is equivalent to her having a winning strategy for the Prompt-LTL game (G, φ_{β}) from v. The latter problem can be solved in doubly-exponential time and a finite-state strategy can effectively be computed [31].

C Proofs from Section 4

Proof (Proof of Lemma 2). We proceed by induction over the structure of φ. The cases of atomic propositions and Boolean connectives are trivial, as they return values from $\mathbb{B}_4$ by definition, provided the arguments are from $\mathbb{B}_4$. Similarly, we have $V^{\mathrm{RD}}(w, [\cdot r \cdot] \varphi) = b_1 b_2 b_3 b_4 \in \mathbb{B}_4$, as the maximization "$b_i = \max\{b'_1, \ldots, b'_i\}$" in the definition enforces the desired monotonicity of the bits b_i.

To conclude, consider a diamond formula $\langle \cdot r \cdot \rangle \varphi$. Applying the induction hypothesis to the tests of r and an induction over the construction of r shows

$$\mathcal{R}_1^{\mathrm{R}}(w, r) \subseteq \mathcal{R}_2^{\mathrm{R}}(w, r) \subseteq \mathcal{R}_3^{\mathrm{R}}(w, r) \subseteq \mathcal{R}_4^{\mathrm{R}}(w, r)$$

for every trace w. Hence, an application of the induction hypothesis for φ yields

$$\max_{j \in \mathcal{R}_1^{\mathrm{R}}(w,r)} V_1^{\mathrm{RD}}(w[j, \infty), \varphi) \leq \max_{j \in \mathcal{R}_2^{\mathrm{R}}(w,r)} V_2^{\mathrm{RD}}(w[j, \infty), \varphi)$$
$$\leq \max_{j \in \mathcal{R}_3^{\mathrm{R}}(w,r)} V_3^{\mathrm{RD}}(w[j, \infty), \varphi) \leq \max_{j \in \mathcal{R}_4^{\mathrm{R}}(w,r)} V_4^{\mathrm{RD}}(w[j, \infty), \varphi)$$

for every trace w. Hence, $V^{\mathrm{RD}}(w, \langle \cdot r \cdot \rangle \varphi) \in \mathbb{B}_4$.

Next, we prove Theorem 3. To this end, in Section C.1, we first recall how to translate guards r into finite non-deterministic automata with special features to account for tests. Then, in Section C.2, we present the translation of rLDL into alternating parity automata of linear size, which can then be further transformed into non-deterministic Büchi automata of exponential size and deterministic parity automata of doubly-exponential size. Such automata are needed for solving the model checking problem and the synthesis problem, respectively.

C.1 Translating Guards into Automata

Recall that P is the (finite) set of atomic propositions. An automaton with tests $\mathfrak{G} = (Q, 2^P, q_I, \delta, F, t)$ consists of a finite set Q of states, the alphabet 2^P, an initial state $q_I \in Q$, a transition function $\delta \colon Q \times (2^P \cup \{\varepsilon\}) \to 2^Q$, a set F of final states, and a partial function t, which assigns to states $q \in Q$ an rLDL formula $t(q)$. These should be thought of as the analogue of tests, i.e., if $t(q)$ is defined, then a run visiting q is only successful, if the word that remains to be processed from q onwards satisfies the formula $t(q)$.

We write $q \xrightarrow{a} q'$ if $q' \in \delta(q, a)$ for $a \in 2^P \cup \{\varepsilon\}$. An ε-path π from q to q' in $\mathfrak{G}$ is a sequence $\pi = q_1 \cdots q_k$ of $k \geq 1$ states with $q = q_1 \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} q_k = q'$. Let $t(\pi) = \{t(q_i) \mid 1 \leq i \leq k\}$ denote the set of tests visited by π and let $\Pi(q, q')$ denote the set of all ε-paths from q to q'.

A run of $\mathfrak{G}$ on $w(0) \cdots w(j-1) \in (2^P)^*$ is a sequence $q_0 q_1 \cdots q_j$ of states such that $q_0 = q_I$ and for every j' in the range $0 \leq j' \leq j-1$ there is a state $q'_{j'}$ reachable from q_i via an ε-path $\pi_{j'}$ and such that $q_{j'+1} \in \delta(q'_{j'}, w(j'))$. The run is accepting if there is a $q'_j \in F$ reachable from q_j via an ε-path π_j.

This slightly unusual definition of runs (but equivalent to the standard one) simplifies our reasoning below. Also, the definition is oblivious to the tests assigned by t. To take them into account, we define for $i \in \{1, 2, 3, 4\}$

$$\mathcal{R}_i^{\mathrm{R}}(w, \mathfrak{G}) = \{j \mid \mathfrak{G} \text{ has an accepting run on } w[0, j) \text{ with } \varepsilon\text{-paths } \pi_0, \dots, \pi_j \text{ s.t.}$$
$$V_i^{\mathrm{RD}}(w[j', \infty), \bigwedge t(\pi_{j'})) = 1 \text{ for every } j' \text{ in the range } 0 \leq j' \leq j\}.$$

Here, $\bigwedge t(\pi_{j'})$ is the conjunction of all formulas in $t(\pi_{j'})$.

Every guard (which is just a regular expression with tests) can be turned into an equivalent automaton with tests via a straightforward generalization of the classical Thompson construction turning classical regular expressions into ε-NFA.

Lemma 3 ([13]). *Every guard r can be translated into an automaton with tests $\mathfrak{G}_r$ such that $\mathcal{R}_i^{\mathrm{R}}(w, r) = \mathcal{R}_i^{\mathrm{R}}(w, \mathfrak{G}_r)$ for every $i \in \{1, 2, 3, 4\}$ and with $|\mathfrak{G}_r| \in \mathcal{O}(|r|)$. Furthermore, all final states of $\mathfrak{G}_r$ are terminal, i.e., they have no outgoing transitions.*

The automaton $\mathfrak{G}_r$ is independent of i, as this value only determines how tests are evaluated. These are handled "externally" in the definition of the semantics. Having thus demonstrated how to turn guards into automata, we now demonstrate how do the same for rLDL formulas.

C.2 Translating rLDL into Alternating Automata

In this subsection, we translate rLDL formulas into alternating parity automata, which are known to be translatable into Büchi automata of exponential size. Hence, the linear translation from rLDL to alternating parity automata we are about to present implies the exponential compilation property for rLDL.

An alternating parity automaton $\mathfrak{A} = (Q, \Sigma, q_I, \delta, \Omega)$ consists of a finite set Q of states, an alphabet Σ, an initial state $q_I \in Q$, a transition function $\delta \colon Q \times \Sigma \to \mathcal{B}^+(Q)$, and a coloring $\Omega \colon Q \to \mathbb{N}$ of the states. Here, $\mathcal{B}^+(Q)$ denotes the set of positive Boolean combinations over Q, which contains in particular the formulas $\mathtt{tt}$ (true) and $\mathtt{ff}$ (false).

A run of $\mathfrak{A}$ on $w = w(0)w(1)w(2) \cdots \in \Sigma^\omega$ is a directed graph $\rho = (V, E)$ where $V \subseteq Q \times \mathbb{N}$ and $((q, n), (q', n')) \in E$ implies $n' = n + 1$, such that $(q_I, 0) \in V$, and such that for all $(q, n) \in V$ we have $\mathrm{Succ}_\rho(q, n) \models \delta(q, w(n))$. Here $\mathrm{Succ}_\rho(q, n)$ denotes the set of successors of (q, n) in ρ projected to Q. A run ρ is accepting if all infinite paths (projected to Q) through ρ satisfy the (max) parity condition, i.e., the maximal color occurring infinitely often on the path is even. The language $L(\mathfrak{A})$ contains all $w \in \Sigma^\omega$ that have an accepting run of $\mathfrak{A}$.

Alternating parity automata are easily seen to be closed under all Boolean operations. Fix automata $\mathfrak{A}_0 = (Q_0, \Sigma, q_I^0, \delta_0, \Omega_0)$ and $\mathfrak{A}_1 = (Q_1, \Sigma, q_I^1, \delta_1, \Omega_1)$.

- $(Q_0, \Sigma, q_I^0, \overline{\delta_0}, \overline{\Omega})$ recognizes $\Sigma^\omega \setminus L(\mathfrak{A}_0)$, where $\overline{\Omega}(q) = \Omega(q) + 1$ and where $\overline{\delta_0}$ is the dual of δ_0, i.e., $\overline{\delta_0}(q, A)$ is obtained from $\delta_0(q, A)$ by replacing each disjunction by a conjunction, each conjunction by a disjunction, each $\mathtt{tt}$ by $\mathtt{ff}$, and each $\mathtt{ff}$ by $\mathtt{tt}$.
- The disjoint union of $\mathfrak{A}_0$ and $\mathfrak{A}_1$ with a fresh initial state q_I and $\delta(q_I, A) = \delta_0(q_I^0, A) \wedge \delta_1(q_I^1, A)$ recognizes $L(\mathfrak{A}_0) \cap L(\mathfrak{A}_1)$.
- The disjoint union of $\mathfrak{A}_0$ and $\mathfrak{A}_1$ with a fresh initial state q_I and $\delta(q_I, A) = \delta_0(q_I^0, A) \vee \delta_1(q_I^1, A)$ recognizes $L(\mathfrak{A}_0) \cup L(\mathfrak{A}_1)$.

The latter two constructions can obviously be generalized to unions and intersections of arbitrary arity while still only requiring a single fresh state.

We prove the following lemma, which implies Theorem 3, as alternating parity automata can be translated into non-deterministic Büchi automata of exponential size.

Lemma 4. *For every rLDL formula φ and every $\beta \in \mathbb{B}_4$, there is an alternating parity automaton $\mathfrak{A}_{\varphi, \beta}$ with $\mathcal{O}(|\varphi|)$ states recognizing the language $\{w \in (2^P)^\omega \mid V^{\mathrm{RD}}(w, \varphi) \succeq \beta\}$.*

Proof. We first construct the desired automaton by structural induction over the construction of φ. Then, we estimate its size. We begin by noting that $\mathfrak{A}_{\varphi,0000}$ is trivial for every formula φ, as it has to accept every input. Hence, we only consider $\beta \succ 0000$ in the remainder of the proof.

For an atomic proposition $p \in P$, $\mathfrak{A}_{p,\beta}$ for $\beta \succ 0000$ is an automaton that accepts exactly those w with $p \in w(0)$. Such an automaton can easily be constructed.

Now, consider a negation $\varphi = \neg\varphi'$: by definition, we have $V^{\mathrm{RD}}(w,\varphi) = 0000$ if $V^{\mathrm{RD}}(w,\varphi') = 1111$, and $V^{\mathrm{RD}}(w,\varphi) = 1111$ if $V^{\mathrm{RD}}(w,\varphi') \neq 1111$. Thus, $\mathfrak{A}_{\varphi,\beta}$ for $\beta \succ 0000$ has to accept the language $\{w \mid V^{\mathrm{RD}}(w,\varphi') \neq 1111\}$, which is the complement of the language of $\mathfrak{A}_{\varphi',1111}$.

Next, consider a conjunction $\varphi = \varphi_0 \wedge \varphi_1$. To this end, recall that $V^{\mathrm{RD}}(w,\varphi) = \min\{V^{\mathrm{RD}}(w,\varphi_0), V^{\mathrm{RD}}(w,\varphi_1)\}$. Hence, $\mathfrak{A}_{\varphi,\beta}$ has to recognize the language

$$\bigcup_{\substack{\beta_0,\beta_1 \in \mathbb{B}_4 \\ \min\{\beta_0,\beta_1\} \succeq \beta}} L(\mathfrak{A}_{\varphi_0,\beta_0}) \cap L(\mathfrak{A}_{\varphi_1,\beta_1}).$$

Such an automaton can be constructed using the closure operations described above.[7]

The construction for a disjunction $\varphi = \varphi_0 \vee \varphi_1$ is dual to the conjunction: we have $V^{\mathrm{RD}}(w,\varphi) = \max\{V^{\mathrm{RD}}(w,\varphi_0), V^{\mathrm{RD}}(w,\varphi_1)\}$ and thus construct $\mathfrak{A}_{\varphi,\beta}$ such that it recognizes the language

$$\bigcup_{\substack{\beta_0,\beta_1 \in \mathbb{B}_4 \\ \max\{\beta_0,\beta_1\} \succeq \beta}} L(\mathfrak{A}_{\varphi_0,\beta_0}) \cap L(\mathfrak{A}_{\varphi_1,\beta_1}).$$

For an implication $\varphi = \varphi_0 \to \varphi_1$, we again implement the semantics via Boolean combinations of automata. Recall that $V^{\mathrm{RD}}(w,\varphi_0 \to \varphi_1)$ is equal to 1111 if $V^{\mathrm{RD}}(w,\varphi_0) \preceq V^{\mathrm{RD}}(w,\varphi_1)$. Otherwise, it is equal to $V^{\mathrm{RD}}(w,\varphi_1)$. Hence, we construct $\mathfrak{A}_{\varphi,\beta}$ so that it recognizes the language

$$\left(\bigcup_{\substack{\beta_0,\beta_1 \in \mathbb{B}_4 \\ \beta_0 \preceq \beta_1}} L(\mathfrak{A}_{\varphi_0,\beta_0}) \cap L(\mathfrak{A}_{\varphi_1,\beta_1})\right) \cup \left(\bigcup_{\substack{\beta_0,\beta_1 \in \mathbb{B}_4 \\ \beta_0 \succ \beta_1 \preceq \beta}} L(\mathfrak{A}_{\varphi_0,\beta_0}) \cap L(\mathfrak{A}_{\varphi_1,\beta_1})\right).$$

The left part covers all cases in which the implication evaluates to 1111. Due to $1111 \succeq \beta$ for every β, this part is equal for all automata. The right part covers all other cases, which depend on β.

Now, we turn to the constructions for the guarded temporal operators, which are more involved as we have to combine automata for guards, for the tests occurring in them, and for formulas. We follow the general construction presented by Faymonville and Zimmermann [13], but generalize it to deal with the richer truth values underlying the robust semantics.

First, we consider a diamond formula $\varphi = \langle \cdot r \cdot \rangle \varphi'$ with tests $\theta_1, \ldots, \theta_n$ in r. Recall that we have $V^{\mathrm{RD}}(w,\varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{j \in \mathcal{R}_i^{\mathrm{R}}(w,r)} V_i^{\mathrm{RD}}(w[j,\infty),\varphi')$ for all $i \in \{1,2,3,4\}$. Thus, $\mathfrak{A}_{\varphi,\beta}$ has to accept w if and only if w has an r-match of degree β that is φ'-satisfying of degree β.

By induction hypothesis, we have automata $\mathfrak{A}_{\varphi',\beta}$ and $\mathfrak{A}_{\theta_j,\beta}$ for every test θ_j in r. Also, we have an ε-NFA with tests $\mathfrak{G}_r$ equivalent to r due to Lemma 3. We combine these automata to the alternating automaton $\mathfrak{A}_{\varphi,\beta}$ by non-deterministically guessing a (finite) run of $\mathfrak{G}_r$. Whenever the run encounters a final state, the automaton may jump to the initial state of $\mathfrak{A}_{\varphi',\beta}$ and then behave like that automaton. Furthermore, while simulating $\mathfrak{G}_r$, $\mathfrak{A}_{\varphi,\beta}$ also has to verify that the tests occurring along the guessed run of $\mathfrak{G}_r$ hold true by universally spawning copies of $\mathfrak{A}_{\theta_j,\beta}$ each time a transition state labeled with θ_j is traversed. Since we do not allow for ε-transitions in alternating automata, we have to eliminate the ε-transitions of $\mathfrak{G}_r$ during the construction of $\mathfrak{A}_{\varphi,\beta}$. Finally, in order to prevent $\mathfrak{A}_{\varphi,\beta}$ from simulating $\mathfrak{G}_r$ ad infinitum, the states copied from $\mathfrak{G}_r$ are assigned an odd color, which forces the jump to $\mathfrak{A}_{\varphi,\beta}$ to be executed eventually.

Formally, we define $\mathfrak{A}_{\varphi,\beta} = (Q, 2^P, q_I, \delta, \Omega)$ where

- Q is the disjoint union of the sets of states of the automata $\mathfrak{G}_r$, $\mathfrak{A}_{\theta_j,\beta}$ for $j \in \{1,\ldots,n\}$, and $\mathfrak{A}_{\varphi',\beta}$,
- q_I is the initial state of $\mathfrak{G}_r$,

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[7] The description of the language (and thus the automaton) can be simplified by exploiting the fact that $\beta \preceq \beta'$ implies $\mathfrak{A}_{\varphi,\beta} \subseteq \mathfrak{A}_{\varphi,\beta'}$ for every φ. The same is true for disjunction and implication.

– Ω coincides with the colorings of the automata $\mathfrak{A}_{\theta_j,\beta}$ and $\mathfrak{A}_{\varphi',\beta}$ on their states and assigns color 1 to every state of $\mathfrak{G}_r$,

and where δ is defined as follows: if q is a state of $\mathfrak{G}_r$, then

$$\delta(q,A) = \begin{cases} \bigvee_{q' \in Q^r} \bigvee_{\pi \in \Pi(q,q')} \bigvee_{p \in \delta^r(q',A)} (p \wedge \bigwedge_{\theta_j \in t(\pi)} \delta^j(q_I^j, A)) \\ \vee \\ \bigvee_{q' \in F^r} \bigvee_{\pi \in \Pi(q,q')} (\delta'(q_I', A) \wedge \bigwedge_{\theta_j \in t(\pi)} \delta^j(q_I^j, A)) \end{cases}$$

where q_I^j and q_I' are the initial states of $\mathfrak{A}_{\theta_j,\beta}$ and $\mathfrak{A}_{\varphi',\beta}$, respectively, where Q^r (F^r) is the set of (final) states of $\mathfrak{G}_r$, where δ_r, δ', and δ_j are the transition functions of $\mathfrak{G}_r$, $\mathfrak{A}_{\varphi',\beta}$, and $\mathfrak{A}_{\theta_j,\beta}$ respectively, and where the sets $\Pi(q,q')$ of ε-paths are induced by $\mathfrak{G}_r$. Furthermore, for states q of $\mathfrak{A}_{\varphi',\beta}$, we define $\delta(q,A) = \delta'(q,A)$ and for states q of $\mathfrak{A}_{\theta_j,\beta}$ we define $\delta(q,A) = \delta^j(q,A)$. The resulting automaton accepts w if and only if w has at least one r-match of degree β that is φ'-satisfying of degree β (cf. [13]).

Finally, we consider the box operator, which requires the most involved construction due to the case distinction defining the b_i' and the subsequent maximization to obtain the b_i. First, recall that the semantics of the box operator is not dual to the semantics of the diamond operator. Nevertheless, the dual construction of the one for the diamond operator is useful as a building block. We first present this construction before tackling the construction for the box operator.

In the dual construction, one interprets $\mathfrak{G}_r$ as a universal automaton whose transitions are ignored if the test on the source of the transition fails. Furthermore, each visit to a final state spawns a copy of the automaton $\mathfrak{A}_{\varphi',\beta}$, as every r-match has to be φ'-satisfying. Thus, the states of $\mathfrak{G}_r$ are now accepting, as all r-matches have to be considered, and the automata for the tests are dualized in order to check for the failure of the test.

Formally, this approach yields the alternating parity automaton $(Q, 2^P, q_I, \delta, \Omega)$ where Q and q_I are as above, where

$$\delta(q,A) = \begin{cases} \bigwedge_{q' \in Q^r} \bigwedge_{\pi \in \Pi(q,q')} \bigwedge_{p \in \delta^r(q',A)} (p \vee \bigvee_{\theta_j \in t(\pi)} \delta^j(q_I^j, A)) \\ \wedge \\ \bigwedge_{q' \in F^r} \bigwedge_{\pi \in \Pi(q,q')} (\delta'(q_I', A) \vee \bigvee_{\theta_j \in t(\pi)} \delta^j(q_I^j, A)) \end{cases}$$

for states q of $\mathfrak{G}_r$, where q_I^j and q_I' are the initial states of $\mathfrak{A}_{\theta_j,\beta}$ and $\mathfrak{A}_{\varphi',\beta}$, respectively, where $\delta(q,A) = \delta'(q,A)$ for states q of $\mathfrak{A}_{\varphi',\beta}$, and where $\delta(q,A) = \overline{\delta^j}(q,A)$ for states q of $\mathfrak{A}_{\theta_j,\beta}$. Here, we use the fact that the final states of $\mathfrak{G}_r$ have no outgoing transitions, which implies that no match is missed by contracting an ε-path. Finally, states from $\mathfrak{G}_r$ have color 0, states from $\mathfrak{A}_{\varphi',\beta}$ keep their color, and the colors from the automata $\mathfrak{A}_{\theta_j,\beta}$ are incremented by one. Recall that dualizing the transition relation and incrementing the colors of the automata $\mathfrak{A}_{\theta_j,\beta}$ amounts to complementation. This allows terminating runs of $\mathfrak{G}_r$ if a test does not hold true. The resulting automaton accepts a trace if and only if every r-match of degree β is φ-satisfying of degree β (cf. [13]).

Now, we fix $\varphi = [\cdot r \cdot] \varphi'$. Recall that we have $V^{\mathrm{RD}}(w, \varphi) = b_1 b_2 b_3 b_4$ with $b_i = \max\{b_1', \ldots, b_i'\}$ for some bits b_i'. The maximization is easily implemented using the Boolean closure properties of alternating automata provided we have automata checking that some bit b_i' is equal to one. Two cases are trivial: Indeed, we have $b_1' = 1$ if and only if every r-match of degree 1111 is φ-satisfying of degree 1111. This property is checked by the dual automaton constructed above. Furthermore, $b_4' = 1$ if and only if $V^{\mathrm{RD}}(w, \langle \cdot r \cdot \rangle \varphi') \succeq 0001$ or if there is no r-match of degree 0001. The former language is recognized by $\mathfrak{A}_{\langle \cdot r \cdot \rangle \varphi', 0001}$, the latter one by an automaton we construct below. We then combine these two automata to obtain $\mathfrak{A}_{\varphi, 0001}$.

Hence, it remains to consider b_2' and b_3', which are both defined by a case distinction over the number of r-matches of the trace. These case distinctions are implemented using alternation. To this end, we first show how to test for the three cases, i.e., we argue that the following languages are recognizable by alternating parity automata, where $i \in \{1, 2, 3, 4\}$:

1. $L_i^{\emptyset}(r) = \{w \in (2^P)^{\omega} \mid |\mathcal{R}_i^{\mathrm{R}}(w,r)| = 0\}$.
2. $L_i^f(r) = \{w \in (2^P)^{\omega} \mid 0 < |\mathcal{R}_i^{\mathrm{R}}(w,r)| < \infty\}$.
3. $L_i^{\infty}(r) = \{w \in (2^P)^{\omega} \mid |\mathcal{R}_i^{\mathrm{R}}(w,r)| = \infty\}$.

Let $\theta_1, \ldots, \theta_n$ be the tests in r. By induction hypothesis, we have alternating parity automata $\mathfrak{A}_{\theta_j, \beta}$ for every θ_j and every truth value β.

The first case is already solved, as we have $\mathcal{R}_i^{\mathrm{R}}(w, r) = \emptyset$ if and only if $V_i^{\mathrm{RD}}(w, \langle \cdot r \cdot \rangle \mathtt{tt}) = 0$, which is in turn equivalent to $V^{\mathrm{RD}}(w, \langle \cdot r \cdot \rangle \mathtt{tt}) \prec 0^{i-1}1^{5-i}$, i.e., the complement of the automaton $\mathfrak{A}_{\langle \cdot r \cdot \rangle \mathtt{tt}, 0^{i-1}1^{5-i}}$ recognizes $L_i^{\emptyset}(r)$.

Next, we construct an automaton for the language $L_i^{\infty}(r)$. Then, the automaton for $L_i^f(r)$ is obtained as the intersection of the complement automata for the other two languages (for the given r and i). Thus, we need to construct an automaton that accepts w if there are infinitely many r-matches of degree $0^{i-1}1^{5-i}$ or greater.

The construction of an automaton for $L_i^{\infty}(r)$ is more involved than the previous one, as the automaton $\mathfrak{G}_r$ checking for matches with r is non-deterministic. Nevertheless, we show that standard arguments about such automata still yield the desired result. We say that an infinite sequence $q_0 q_1 q_2 \cdots$ of states is an (infinity) witness for w if q_0 is the initial state of $\mathfrak{G}_r$ and if for every j, there is an accepting run of $\mathfrak{G}_r$ on some prefix of w such that the tests on the associated ε-paths hold w.r.t. V_i^{RD}, and such that $q_0 \cdots q_j$ is a prefix of this run. An application of König's Lemma shows that $\mathcal{R}_i^{\mathrm{R}}(w, r)$ is infinite if and only if $\mathfrak{G}_r$ has a witness for w.

Thus, the automaton recognizing $L_i^{\infty}(r)$ has to find such a witness for w while processing w. This is implemented as follows: we start with $\mathfrak{G}_r$, eliminate ε-transitions non-deterministically as above and spawn a copy of $\mathfrak{A}_{\theta_j, 0^{i-1}1^{5-i}}$ when traversing a state with test θ_j. Furthermore, every time a letter is processed, another copy of $\mathfrak{G}_r$ is spawned (with a disjoint set of states). The coloring of the original copy is chosen such that the automaton has to process infinitely many letters and such that the disjoint copies have to reach a final state of $\mathfrak{G}_r$. Hence, the resulting automaton accepts w if and only if there is a witness for w. We leave the details to the industrious reader and just note that we have now constructed all automata we need to capture the cases in the case distinction.

Extending the construction just presented also allows us to construct an automaton that accepts a trace w if and only if it has infinitely many φ'-satisfying r-matches (both of degree $0^{i-1}1^{5-i}$). To this end, the copies spawned to check for matches do not terminate in an accepting sink, but instead spawn a copy of $\mathfrak{A}_{\varphi', 0^{i-1}1^{5-i}}$ to check for satisfaction of φ'. Similarly, we can construct an automaton that accepts a trace w if and only if it has infinitely many r-matches of degree $0^{i-1}1^{5-i}$ that are *not* φ-satisfying of degree $0^{i-1}1^{5-i}$. Again, we leave the details to the reader.

These automata also allow us to construct an automaton that accepts a trace w if and only if $\mathcal{R}_i^{\mathrm{R}}(w, r)$ is infinite and almost all r-matches in $\mathcal{R}_i^{\mathrm{R}}(w, r)$ are φ'-satisfying (both of degree $0^{i-1}1^{5-i}$). This automaton is obtained by taking the automaton checking for infinitely many φ'-satisfying r-matches (both of degree β) and intersecting it with the complement of the one checking for infinitely many r-matches that are not φ-satisfying of degree $0^{i-1}1^{5-i}$.

Combining the automata checking the cases of the case distinction with the automata checking for φ'-satisfiability yields the desired automata for b_2' and b_3': A case distinction is easily implemented using the Boolean closure properties and all necessary auxiliary automata have been constructed above.

It remains to argue that $\mathfrak{A}_{\varphi, \beta}$ is of linear size in $|\varphi|$. To this end, we say that an alternating parity automaton $(Q', \Sigma, q_I', \delta', \Omega')$ is a subautomaton of $(Q, \Sigma, q_I, \delta, \Omega)$ if $Q' \subseteq Q$, $\delta'(q, A) = \delta(q, A)$ for every $q \in Q'$ and every $A \in \Sigma$, and $\Omega'(q) = \Omega(q)$ for every $q \in Q'$.

Inspecting the construction above shows that an automaton $\mathfrak{A}_{\varphi, \beta}$ is built from automata for immediate subformulas (w.r.t. all truth values if necessary), a test automaton (if applicable), and a constant number of fresh states. Furthermore, if formulas share subformulas, then the construction can also share these subautomata. Hence, we obtain the desired linear upper bound on the size of $\mathfrak{A}_{\varphi, \beta}$.

It is not straightforward that the equivalent Büchi automata as in Theorem 3 can be constructed efficiently, as the definition of the alternating automaton involves ε-paths of arbitrary length. However, these can be restricted to simple paths, which are of bounded length. Then, as it is done for the similar construction for PLDL [13], one can show that the Büchi automata can be constructed on-the-fly in polynomial space. This is sufficient for our applications later on.

Furthermore, as non-deterministic Büchi automata can be translated into deterministic parity automata (see, e.g., [15] for definitions), we obtain the following corollary of Theorem 3.

Corollary 1. *Let φ be an rLDL formula, $n = |\varphi|$, and $\beta \in \mathbb{B}_4$. There is a deterministic parity automaton $\mathfrak{P}_{\varphi,\beta}$ with $2^{2^{\mathcal{O}(n \log n)}}$ states and with $2^{\mathcal{O}(n \log n)}$ colors recognizing the language $\{w \in (2^P)^\omega \mid V^{\mathrm{RD}}(w, \varphi) \succeq \beta\}$.*

D Proofs from Section 4.1

Proof (Proof of Theorem 4). Let us first embed rLTL$(\boxdot, \Diamonddot)$ into rLDL by showing that the syntactic embedding of LTL into LDL extends to robust semantics. Recall that rLTL$(\boxdot, \Diamonddot)$ only has temporal operators $\Diamonddot$ and $\boxdot$, which we replace by $\langle \mathtt{tt}^* \rangle$ and $[\cdot\mathtt{tt}^*\cdot]$. Note that $\mathcal{R}_i^{\mathrm{R}}(w, \mathtt{tt}^*) = \mathbb{N}$ holds true for every w and every i. Hence, a straightforward induction shows that the resulting rLDL formula is equivalent to the original rLTL$(\boxdot, \Diamonddot)$ formula. In particular, only the first case in the case distinctions defining the semantics of the robust box operator of rLDL is used, which mimics the definition of the semantics of the always operator in rLTL$(\boxdot, \Diamonddot)$.

Conversely, using a straightforward induction over the structure of LDL formulas, we can show that $V^{\mathrm{D}}(w, \varphi) = V_1^{\mathrm{RD}}(w, \varphi')$ for every w and every LDL formula φ, where φ' is the rLDL formula obtained from φ by replacing each $\langle r \rangle$ with $\langle \cdot r \cdot \rangle$, each $[r]$ with $[\cdot r \cdot]$, and each implication $\psi_1 \to \psi_2$ with $\neg\psi_1 \vee \psi_2$. This shows that LDL can be embedded into rLDL. Note, however, that we need to replace each implication with a negation and a disjunction. This is necessary to account for the more complex definition of implications in rLTL$(\boxdot, \Diamonddot)$/rLDL.

Proof (Proof of Theorem 5). As argued above, we only have to consider the direction from rLDL to LDL. Hence, fix an rLDL formula φ and $i \in \{1, 2, 3, 4\}$. Due to Theorem 3, $\{w \in (2^P)^\omega \mid V_i^{\mathrm{RD}}(w, \varphi) = 1\}$ is ω-regular. Hence, due to LDL being equi-expressive to the ω-regular languages [27], there is also an LDL formula φ_i with $V^{\mathrm{D}}(w, \varphi_i) = 1$ if and only if $V_i^{\mathrm{RD}}(w, \varphi) = 1$. Hence, φ_i has the desired properties.

Let us analyze the complexity of the translation in more detail. A non-deterministic Büchi automaton for an rLDL formula is in general of exponential size and has to be determinized before it can be translated into LDL (say with max-parity acceptance), which incurs a second exponential blowup. The resulting deterministic automaton can then be translated into LDL with an exponential blowup. The resulting formula expresses that the unique run on the input satisfies the following property: there is an even color c and a position j such that after j no larger color appears, c appears at least once, and every time color c appears, it is not the last occurrence of c (see [29] for a similar construction). Both properties are easily expressed in LDL by constructing guards $r_{q,q'}$ that match infixes that take the parity automaton from q to q'. The number of subformulas is polynomial, but the guards $r_{q,q'}$ have in general exponential size (both measured in the size of the doubly-exponential deterministic automaton). Hence, the full construction incurs a triply-exponential blowup. It is open whether this is unavoidable.

E Proofs from Section 4.2

Using the translation of rLDL formulas to alternating parity automata and subsequently to non-deterministic Büchi automata, Problem 3 can be solved as follows:

1. Translate the transition system $\mathcal{S}$ into a non-deterministic Büchi automaton $\mathfrak{B}_{\mathcal{S}}$ with $L(\mathfrak{B}_{\mathcal{S}}) = \{\lambda(\rho) \in (2^P)^\omega \mid \rho \in \Pi_{\mathcal{S}}\}$ in the usual way: $\mathfrak{B}_{\mathcal{S}}$ has the same states as $\mathcal{S}$, the transitions are $\{(s, \lambda(s), s') \mid (s, s') \in E\}$, and all states are accepting.
2. Construct the alternating parity automaton $\mathfrak{A}_{\varphi,\beta}$ accepting the language $\{w \in (2^P)^\omega \mid V^{\mathrm{RD}}(w, \varphi) \succeq \beta\}$.
3. Complement $\mathfrak{A}_{\varphi,\beta}$ to obtain an alternating parity automaton $\overline{\mathfrak{A}_{\varphi,\beta}}$ accepting the language $\{w \in (2^P)^\omega \mid V^{\mathrm{RD}}(w, \varphi) \prec \beta\}$.
4. Convert $\overline{\mathfrak{A}_{\varphi,\beta}}$ into an equivalent non-deterministic Büchi automaton $\overline{\mathfrak{B}_{\varphi,\beta}}$ and compute the product automaton $\mathfrak{B}$ with $L(\mathfrak{B}) = L(\mathfrak{B}_{\mathcal{S}}) \cap L(\overline{\mathfrak{B}_{\varphi,\beta}})$ in the usual way.
5. Check whether $L(\mathfrak{B}) = \emptyset$ using a standard algorithm such as a nested depth-first search [4]. The answer to Problem 3 is "yes" if and only if $L(\mathfrak{B}) = \emptyset$.

The number of states of the alternating parity automata in Step 2 and 3 is both in $\mathcal{O}(|\varphi|)$. Thus, the number of states of the non-deterministic Büchi automaton $\overline{\mathfrak{B}_{\varphi,\beta}}$ constructed in Step 4 is in $2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}$, and that of $\mathfrak{B}$ is in $|\mathcal{S}|\cdot 2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}$, where $|\mathcal{S}|$ denotes the number of states of the transition system $\mathcal{S}$ (cf. Theorem 3). Finally, the time required for the emptiness check in Step 5 is quadratic in the number of states of $\mathfrak{B}$ (linear in the number of $\mathfrak{B}$'s transitions). Consequently, the rLDL model checking problem can be solved in time $|\mathcal{S}|^2\cdot 2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}$ and, hence, is in ExpTime. The following proof shows that the rLDL model checking problem is in fact PSpace-complete, matching exactly the complexity class of the model checking problem for LDL.

Proof (Proof of Theorem 6). As shown above, the rLDL model checking problem is in ExpTime. To show membership in PSpace, we use the observation that given two states of $\mathfrak{B}$, one can decide in polynomial space whether the second state is a successor of the first one (cf. Vardi and Wolper [28]). Moreover, one can represent states of $\mathfrak{B}$ in polynomial space. This allows running the classical model checking algorithm, which searches for a counterexample, in polynomial space by guessing an appropriate run. PSpace-hardness, on the other hand, follows immediately from the facts that (a) the LTL semantics is embedded in rLDL, via the embedding of LDL in rLDL (see Theorem 5), and (b) LTL model checking is PSpace-hard [23]. Thus, rLDL model checking is PSpace-hard as well.

Before we move on to reactive synthesis, let us briefly remark that the model checking problem for rLTL($\Box, \Diamond$) is defined slightly differently. Instead of asking whether $V^{\mathrm{R}}(\lambda(\rho), \varphi) \succeq \beta$, Tabuada and Neider [26] fix a set $B \subseteq \mathbb{B}_4$ and ask whether $V^{\mathrm{R}}(\lambda(\rho), \varphi) \in B$ for all paths $\rho \in \Pi_{\mathcal{S}}$. However, this slightly more general problem can easily be answered by a simple adaptation of Step 2 of the procedure above: given a (finite) set $B \subseteq \mathbb{B}_4$, we construct an alternating parity automaton accepting the language $\{w \in (2^P)^\omega \mid V^{\mathrm{RD}}(w, \varphi) \in B\}$ using Boolean combinations of the automata $\mathfrak{A}_{\varphi,\beta}$. Then, it is not hard to verify that this variant of the rLDL model checking problem is also PSpace-complete.

Proof (Proof of Theorem 7). We proceed in several steps constructing a reduction to a parity game.

1. Construct the deterministic parity automaton $\mathfrak{P}_{\varphi,\beta}$ recognizing the language $\{w \in (2^P)^\omega \mid V^{\mathrm{RD}}(w, \varphi) \succeq \beta\}$ according to Corollary 1.
2. Construct the product of $\mathfrak{P}_{\varphi,\beta} = (Q, \Sigma, q_I, \delta, \Omega)$ and the labeled game graph $G = (V_0, V_1, E, \lambda)$. This product is a classical (non-labeled) parity game $\mathcal{G}' = (G', \Omega')$ consisting of a game graph $G' = (V_0', V_1', E', \Omega')$ with $V_0' = V_0 \times Q$, $V_1' = V_1 \times Q$, and $E = \{((v, q), (v', \delta(q, \lambda(v)))) \mid (v, v') \in E\}$ as well as a parity winning condition Ω' with $\Omega'((v, q)) = \Omega(q)$ for each $(v, q) \in V_0' \cup V_1'$. One obtains a play ρ in the original game $\mathcal{G}$ from a play ρ' in the extended game $\mathcal{G}'$ by projecting the vertices of ρ' onto the first component. Thus, Player 0 wins a play ρ' in $\mathcal{G}'$ from a vertex (v, q_I) if and only if the trace $\lambda(\rho)$ obtained from the corresponding play ρ in $\mathcal{G}$ satisfies $V^{\mathrm{RD}}(\lambda(\rho), \varphi) \succeq \beta$.
3. Solve the game $\mathcal{G}'$ with standard algorithms for parity games, e.g., the recent quasi-polynomial time algorithm [6]. Finally, check and return whether Player 0 has a winning strategy from vertex (v, q_I) and return a finite-state winning strategy if so.

The above reduction is a standard game reduction, whose correctness can be shown using standard techniques. In fact, it provides a 2ExpTime algorithm to solve Problem 4: due to Corollary 1, the deterministic parity automaton $\mathfrak{P}_{\varphi,\beta}$ constructed in Step 1 has $2^{2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}}$ states and $2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}$ colors; consequently, the parity game $\mathcal{G}'$ of Step 2 has $|V|\cdot 2^{2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}}$ vertices and $2^{\mathcal{O}(|\varphi|\cdot\log|\varphi|)}$ colors; thus, using the quasi-polynomial algorithm for solving parity games [6] results in a doubly-exponential algorithm for Problem 4.

On the other hand, the fact that rLDL subsumes LDL and, hence, LTL immediately implies that solving rLDL games is 2ExpTime-hard since solving LTL games is already 2ExpTime-hard [22].

F   More on Robust Prompt-LDL

We begin by introducing the semantics of rPrompt-LDL. As this logic is an extension of Prompt-LTL, it features negations only at the level of atomic propositions, and does not allow implications. The formulas of rPrompt-LDL are given be the grammar

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle \cdot r \cdot \rangle\, \varphi \mid [\cdot r \cdot]\, \varphi \mid \langle \cdot r \cdot \rangle_{\mathbf{P}}\, \varphi$$
$$r ::= \phi \mid \varphi? \mid r + r \mid r\,;r \mid r^*$$

where p again ranges over the atomic propositions in P and ϕ over propositional formulas over P. Furthermore, the size of a formula is defined as for LDL and rLDL.

The semantics are defined as expected: We add a bound k to the semantics of rLDL, which bounds the scope of the prompt diamond operator.

- $V^{\mathrm{RPD}}(w, k, p) = \begin{cases} 1111 & \text{if } p \in w(0), \\ 0000 & \text{if } p \notin w(0), \end{cases}$

- $V^{\mathrm{RPD}}(w, k, \neg p) = \begin{cases} 0000 & \text{if } p \in w(0), \\ 1111 & \text{if } p \notin w(0), \end{cases}$

- $V^{\mathrm{RPD}}(w, k, \varphi_0 \wedge \varphi_1) = \min\{V^{\mathrm{RPD}}(w, k, \varphi_0), V^{\mathrm{RPD}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{RPD}}(w, k, \varphi_0 \vee \varphi_1) = \max\{V^{\mathrm{RPD}}(w, k, \varphi_0), V^{\mathrm{RPD}}(w, k, \varphi_1)\}$,
- $V^{\mathrm{RPD}}(w, k, \langle \cdot r \cdot \rangle \varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{j \in \mathcal{R}_i^{\mathrm{RP}}(w, k, r)} V_i^{\mathrm{RPD}}(w[j, \infty), k, \varphi)$,
- $V^{\mathrm{RPD}}(w, k, [\cdot r \cdot] \varphi) = b_1 b_2 b_3 b_4$ with $b_i = \max\{b_1', \ldots, b_i'\}$ for every $i \in \{1, 2, 3, 4\}$, where
  - $b_1' = \min_{j \in \mathcal{R}_1^{\mathrm{RP}}(w, k, r)} V_1^{\mathrm{RPD}}(w[j, \infty), k, \varphi)$,
  - $b_2' = \begin{cases} \max_{j' \in \mathbb{N}} \min_{j \in \mathcal{R}_2^{\mathrm{RP}}(w, k, r) \cap \{j', j'+1, j'+2, \ldots\}} V_2^{\mathrm{RPD}}(w[j, \infty), k, \varphi) & \text{if } |\mathcal{R}_2^{\mathrm{RP}}(w, k, r)| = \infty, \\ \min_{j \in \mathcal{R}_2^{\mathrm{RP}}(w, k, r)} V_2^{\mathrm{RPD}}(w[j, \infty), k, \varphi) & \text{if } 0 < |\mathcal{R}_2^{\mathrm{RP}}(w, k, r)| < \infty, \\ 1 & \text{if } |\mathcal{R}_2^{\mathrm{RP}}(w, k, r)| = 0, \end{cases}$
  - $b_3' = \begin{cases} \min_{j' \in \mathbb{N}} \max_{j \in \mathcal{R}_3^{\mathrm{RP}}(w, k, r) \cap \{j', j'+1, j'+2, \ldots\}} V_3^{\mathrm{RPD}}(w[j, \infty), k, \varphi) & \text{if } |\mathcal{R}_3^{\mathrm{RP}}(w, k, r)| = \infty, \\ \max_{j \in \mathcal{R}_3^{\mathrm{RP}}(w, k, r)} V_3^{\mathrm{RPD}}(w[j, \infty), k, \varphi) & \text{if } 0 < |\mathcal{R}_3^{\mathrm{RP}}(w, k, r)| < \infty, \\ 1 & \text{if } |\mathcal{R}_3^{\mathrm{RP}}(w, k, r)| = 0, \end{cases}$
  - $b_4' = \begin{cases} \max_{j \in \mathcal{R}_4^{\mathrm{RP}}(w, k, r)} V_4^{\mathrm{RPD}}(w[j, \infty), k, \varphi) & \text{if } |\mathcal{R}_4^{\mathrm{RP}}(w, k, r)| > 0, \\ 1 & \text{if } |\mathcal{R}_4^{\mathrm{RP}}(w, k, r)| = 0, \end{cases}$ and
- $V^{\mathrm{RPD}}(w, k, \langle \cdot r \cdot \rangle_{\mathbf{p}} \varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{j \in \mathcal{R}_i^{\mathrm{RP}}(w, k, r) \cap \{0, \ldots, k\}} V_i^{\mathrm{RPD}}(w[j, \infty), k, \varphi)$.

Here, we adapt the definition of $\mathcal{R}_i^{\mathrm{R}}$ to account for the parameter k: $\mathcal{R}_i^{\mathrm{RP}}(w, k, \varphi?) = \{0\}$ if $V_i^{\mathrm{RPD}}(w, k, \varphi) = 1$ and $\mathcal{R}_i^{\mathrm{RP}}(w, k, \varphi?) = \emptyset$ otherwise. All other cases are defined as before, but propagate the parameter k.

rLDL (and thus also LDL) and Prompt-LDL are syntactic fragments of rPrompt-LDL, but Prompt-LTL is not, as the semantics of the release operator are not compatible with the semantics of rPrompt-LDL.

It turns out that our model checking and synthesis for rLDL and rPrompt-LTL cannot easily be combined (we discuss the reasons why in Section 5). Thus, we identify a fragment of rPrompt-LDL that restricts the use of regular expression as guards of always and eventually operators: we only allow so-called test-free and limit-matching expression. For this fragment, we show that both the model checking and the synthesis problem are decidable.

F.1 Restricting Guards in rPrompt-LDL

We say that a guard r is test-free if it does not contain tests as atoms, but only propositional formulas over the atomic propositions. A formula is test-free if each of its guards is test-free. In the remainder, we only consider test-free formulas. As the adaptions made to define $\mathcal{R}_i^{\mathrm{RP}}$ are only concerned with tests, they can be ignored when reasoning about test-free formulas.

Remark 1. Let r be a test-free guard. Then, $\mathcal{R}_i^{\mathrm{RP}}(w, k, r)$ is independent of i and k for every trace w.

Hence, in the following, we use $\mathcal{R}(w, r)$ (as defined for LDL) instead of $\mathcal{R}_i^{\mathrm{RP}}(w, k, r)$, since the definitions coincide for test-free guards.

We say that a test-free guard r is limit-matching if we have $|\mathcal{R}(w, r)| = \infty$ for every trace w. This is well-defined due to the previous remark. Again, a test-free formula is limit-matching if each of its guards is limit-matching.

Lemma 5. *The problem "Given a test-free formula φ, is φ limit-matching?" is in* PSPACE.

Proof. The problem is in PSPACE if one can decide in polynomial space whether a single test-free guard is limit-matching. Hence, let r be such a guard, which is limit-matching if and only if infinitely many prefixes of each trace w match r. An application of König's Lemma yields that the latter condition is equivalent to each w being Büchi-accepted by $\mathfrak{G}_r$. Due to test-freeness, $\mathfrak{A}_r$ can indeed be seen as a Büchi

automaton with ε-transitions. Hence, r is limit-matching if and only if $\mathfrak{G}_r$ is universal, which can be decided in polynomial space [24] (after eliminating ε-transitions). The automaton being of the same size as the guard and being efficiently constructible concludes the proof.

Example 5. Recall the formula $\varphi = [\cdot((\neg t)^* \,;\, t \,;\, (\neg t)^* \,;\, t)^* \cdot] \langle \cdot \mathtt{tt}^* \cdot \rangle_{\mathbf{p}} s$ from Example 4. It is test-free, but not limit-matching as traces with finitely many t only have finitely many $((\neg t)^* \,;\, t \,;\, (\neg t)^* \,;\, t)^*$-matches.

Nevertheless, test-free and limit-matching rPrompt-LDL formulas can make use of arbitrary modulo counting, a significant advance in expressiveness over classical LTL, thus witnessing the usefulness of the fragment.

For example, the formula $[\cdot r \cdot] \langle \cdot r \cdot \rangle_{\mathbf{p}} s$ with $r = (\mathtt{tt} \,;\, \mathtt{tt})^*$ expresses, when evaluated with respect to a bound k, that the distance between synchronizations at even positions is bounded by k, i.e., we use the test-free limit-matching guards to "filter out" the odd positions.

Let us note that for LDL, the test-free fragment is of equal expressive power as full LDL, albeit potentially less succinct. This claim follows easily from translating Büchi automata into LDL formulas, which results in test-free formulas.

In the following, we consider the model checking and the synthesis problem for test-free limit-matching formulas. To this end, we proceed as in the case of rPrompt-LTL: We reduce these problems to those for Prompt-LDL, i.e., we present a reduction-based translation to Büchi automata.

Due to only considering limit-matching formulas, we do not have to deal with the cases of having only finitely many matches of a guard. On the other hand, we have to "split" guards to capture the semantics of the robust diamond operator (recall the discussion in Section 5). Here, we exploit the formula under consideration being test-free.

The main technical result on this fragment states that the logic can be derobustified, i.e., translated into Prompt-LDL.

Theorem 8. *For every test-free limit-matching rPrompt-LDL formula φ and every $\beta \in \mathbb{B}_4$, there is a Prompt-LDL formula φ_β such that $V^{\mathrm{RPD}}(w, k, \varphi) \succeq \beta$ if and only if $V^{\mathrm{PD}}(w, k, \varphi_\beta) = 1$.*

Proof. Before we present the translation, we need to explain how to "split" guards, which is necessary to implement the semantics of the robust box operator (recall the discussion in Section 5). For example, we have to check that almost all r-matches are ψ-satisfying for some guard r and some subformula ψ. In LTL, "almost all" is expressed by $\Diamond \Box$. We will use the analogous LDL operators, i.e., a formula of the form $\langle \cdot \rangle [\,\cdot\,]$. But now we need guards r_0 and r_1 for the diamond and the box operator so that the concatenation $r_0 r_1$ is equivalent to r. To this end, we transform r into a deterministic automaton and then have such a pair of guards for every intermediate state that can be reached by the automaton. Ultimately, we then end up with a disjunction of formulas of the form $\langle \cdot \rangle [\,\cdot\,]$.

Let r be a test-free guard. Applying Lemma 3 to r yields an ε-NFA $\mathfrak{G}_r$ without tests. Hence, eliminating ε-transitions and determinizing the resulting automaton yields a deterministic finite automaton $\mathfrak{D}_r$ such that $w[0, j)$ is accepted by $\mathfrak{D}_r$ if and only if $j \in \mathcal{R}(w, r)$. Furthermore, due to test-freeness, acceptance of $w[0, j)$ by $\mathfrak{D}_r$ only depends on the prefix $w[0, j)$ of w, but not on the corresponding suffix $w[j, \infty)$. This property, which underlies the following construction, does not hold true for guards with tests.

Now, let Q be the set of states of $\mathfrak{D}_r$, q_I the initial state, and F the set of final states. Then, one can efficiently construct regular expressions (i.e., guards) $r_{q_I, q}$ and $r_{q, F}$ such that $w \in (2^P)^*$ is in the language of $r_{q_I, q}$ (of $r_{q, F}$) if and only if the unique run of $\mathfrak{D}_r$ starting in q_I (in q) ends in q (in F).

Now, we are ready to construct φ_β. Again, the case $\beta = 0000$ is trivial. Hence, we assume $\beta \succ 0000$ in the following. We proceed by induction over the construction of the formula:

- $p_\beta = p$ and $(\neg p)_\beta = \neg p$ for all atomic propositions $p \in P$ and all $\beta \succ 0000$.
- $(\varphi_0 \wedge \varphi_1)_\beta = (\varphi_0)_\beta \wedge (\varphi_1)_\beta$ for all $\beta \succ 0000$.
- $(\varphi_0 \vee \varphi_1)_\beta = (\varphi_0)_\beta \vee (\varphi_1)_\beta$ for all $\beta \succ 0000$.
- $(\langle \cdot r \cdot \rangle \varphi)_\beta = \langle r \rangle (\varphi_\beta)$ for all $\beta \succ 0000$
- $([\cdot r \cdot] \varphi)_{1111} = [r] (\varphi_{1111})$,
- $([\cdot r \cdot] \varphi)_{0111} = \bigvee_{q \in Q} \langle r_{q_I, q} \rangle [r_{q, F}] (\varphi_{0111})$, where $\mathfrak{D}_r = (Q, 2^P, q_I, \delta, F)$,
- $([\cdot r \cdot] \varphi)_{0011} = \bigwedge_{q \in Q} [r_{q_I, q}] \langle r_{q, F} \rangle (\varphi_{0011})$, where $\mathfrak{D}_r = (Q, 2^P, q_I, \delta, F)$,
- $([\cdot r \cdot] \varphi)_{0001} = \langle r \rangle (\varphi_{0001})$, and
- $(\langle \cdot r \cdot \rangle_{\mathbf{p}} \varphi)_\beta = \langle r \rangle_{\mathbf{p}} (\varphi_\beta)$ for all $\beta \succ 0000$.

A straightforward induction over the construction of φ, relying on the fact that φ is limit-matching, yields the correctness of the translation. The fact that φ is limit-matching explains the construction of $([\cdot r \cdot] \varphi)_\beta$, which only has to implement the first case ("$|\mathcal{R}(w, r)| = \infty$") of the definition of the semantics.

Now, the model checking and the synthesis problem for rPrompt-LDL, which are defined as expected, can be solved by reducing them to their analogues for Prompt-LDL (cf. Section 3). We obtain the following results.

Corollary 2. *The rPrompt-LDL model checking and synthesis problem are decidable for the test-free limit-matching fragment.*

We refrain from specifying the exact complexity of the algorithms, as we conjecture them to be several exponents away from optimal algorithms: The guards $r_{q_I, q}$ and $r_{q, F}$ are already of doubly-exponential size and we still have to translate the formula φ_β containing these guards into (deterministic) automata to solve the problems.

Note that our approach for the fragment, which relies on a translation to Prompt-LDL, cannot easily be extended to formulas with tests and to formulas with non-limit-matching guards. The existence of tests complicates the construction of the deterministic automaton required to "split" the guards. Consider, for example, the guard $(\varphi_0 ?\,;a\,;\varphi_0') + (\varphi_1 ?\,;a\,;\varphi_1')$: after processing an a, depending on which tests hold true before the a, the automaton still has to distinguish whether φ_0' or φ_1' has to hold after processing the a. Implementing this requires non-determinism that cannot be resolved while only reading a prefix of a trace.

Complicating the situation even further, the lack of negations in prompt logics does not allow to "disambiguate" the guard. Similarly, allowing non-limit-matching guards requires us to implement the full case distinction in the definition of the semantics of the robust box operator. However, implementing a case distinction in Prompt-LDL is again complicated by the lack of negations.