Degrees of Lookahead in Context-free Infinite Games

Joint work with Wladimir Fridman and Christof Löding

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Motivation

Starting points:

- Walukiewicz: Solving games with deterministic context-free winning conditions in exponential time.

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- Walukiewicz: Solving games with deterministic context-free winning conditions in exponential time.
- Hosch & Landweber; Holtmann, Kaiser & Thomas: Delay games with regular winning conditions.

Here: delay games with deterministic context-free winning conditions.

- Algorithmic properties.
- Bounds on delay.

Outline

1. **Definitions**
2. Undecidability Results
3. Lower Bounds on Delay
4. Conclusion

The Delay Game $\Gamma_f(L)$

- Delay function: $f : \mathbb{N} \to \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
- Two players: Input (I) vs. Output (O).

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Example

$L = \{\binom{0}{0}^{n_1}\binom{0}{1}^{n_1}\binom{\$}{\$}\binom{0}{0}^{n_2}\binom{0}{1}^{n_2}\binom{\$}{\$}\cdots \mid n_i > 0\}$ and $f(i) = 2$ for all i

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0 0 0 0 \$ 0
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0	0	0	0	$	0
0	0	1			

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0	0	0	0	\$	0	0	\$
0	0	1					

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0	0	0	0	\$	0	0	\$	0	0
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0	0	0	0	$	0	0	$	0	0	0	0
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0	0	0	0	\$	0	0	\$	0	0	0	0	0	0
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0	0	0	0	\$	0	0	\$	0	0	0	0	0	0	0	0
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0	0	0	0	\$	0	0	\$	0	0	0	0	0	0	0	0
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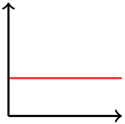
Classifying Delay Functions

1. constant delay function: $f(0) = d$ and $f(i) = 1$ for $i > 0$.

Lookahead
$\sum_{i=0}^{n-1} f(i) - n$

1.)



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1. constant delay function: $f(0) = d$ and $f(i) = 1$ for $i > 0$.
2. linear delay function: $f(i) = k$ for $i \geq 0$.

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$$\sum_{i=0}^{n-1} f(i) - n$$

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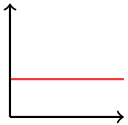
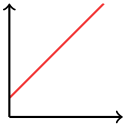


2.)



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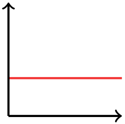
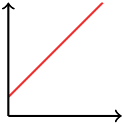
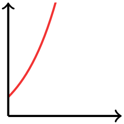
1.) 

2.) 

3.) 

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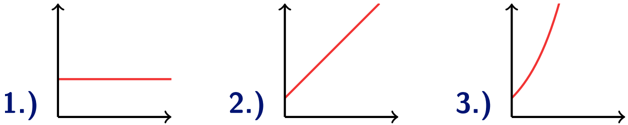

Player O wins the game induced by L with finite (constant, linear, elementary) delay, if there exists an arbitrary (constant, linear, elementary) function f s.t. O has a winning strategy for $\Gamma_f(L)$.

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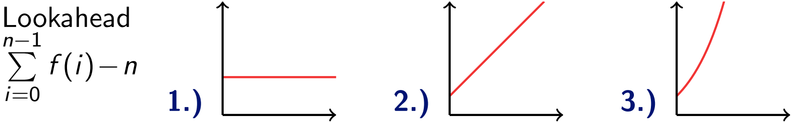


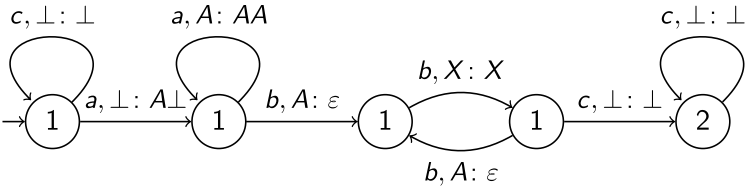
Player O wins the game induced by L with finite (constant, linear, elementary) delay, if there exists an arbitrary (constant, linear, elementary) function f s.t. O has a winning strategy for $\Gamma_f(L)$.

Theorem (HL72, HKT10)

For regular L: Player O wins the game induced by L with finite delay iff she wins it with double-exponential constant delay.

ω-Pushdown Automata

Winning conditions: L recognized by a deterministic ω-pushdown automaton with parity acceptance (parity-DPDA).



Language: $\{c^* a^n b^{2n} c^\omega \mid n > 0\}$.

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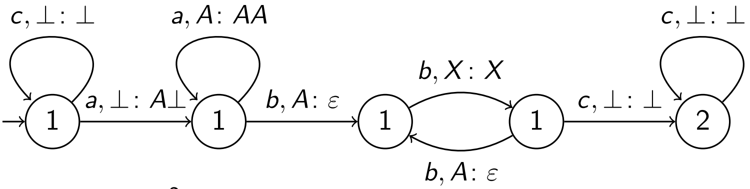

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Restrictions:

- One-counter: just one stack symbol.
- Visibly: $\Sigma = \underbrace{\Sigma_c}_{Push} \cup\!\!\!\!\cdot\;\; \underbrace{\Sigma_r}_{Pop} \cup\!\!\!\!\cdot\;\; \underbrace{\Sigma_s}_{Skip}$.

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2. **Undecidability Results**
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A Decidable Case

Theorem

The following problem is decidable:

Input: *Parity-DPDA $\mathcal{A}$ and f s.t. $\{i \mid f(i) \neq 1\}$ is finite.*

Question: *Does Player O win $\Gamma_f(L(\mathcal{A}))$?*

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- $L' = \{\binom{\alpha(0)}{\$}\binom{\alpha(1)}{\$}\binom{\alpha(2)}{\beta(0)}\binom{\alpha(3)}{\$}\binom{\alpha(4)}{\beta(1)}\binom{\alpha(5)}{\beta(2)}\cdots \mid \binom{\alpha(0)}{\beta(0)}\binom{\alpha(1)}{\beta(1)}\binom{\alpha(2)}{\beta(2)}\cdots \in L(\mathcal{A})\}$.
- L' deterministic context-free.

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The following problem is decidable:

Input: *Parity-DPDA $\mathcal{A}$ and f s.t. $\{i \mid f(i) \neq 1\}$ is finite.*

Question: *Does Player O win $\Gamma_f(L(\mathcal{A}))$?*

Proof Idea

- Suppose $f(0) = 3$, $f(1) = 2$, $f(i) = 1$ for $i > 1$.
- $L' = \{ \binom{\alpha(0)}{\$} \binom{\alpha(1)}{\$} \binom{\alpha(2)}{\beta(0)} \binom{\alpha(3)}{\$} \binom{\alpha(4)}{\beta(1)} \binom{\alpha(5)}{\beta(2)} \cdots \mid \binom{\alpha(0)}{\beta(0)} \binom{\alpha(1)}{\beta(1)} \binom{\alpha(2)}{\beta(2)} \cdots \in L(\mathcal{A}) \}$.
- L' deterministic context-free.
- Now we have a game without delay.
- Apply Walukiewicz's Theorem: Games with deterministic context-free winning conditions can be solved effectively.

Undecidability

Theorem

The following problem is undecidable:

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Proof Idea

Preliminaries:

- Reduction from halting problem for 2-register machines.
- Encode configuration (ℓ, n_0, n_1) by $\ell a^{n_0} b^{n_1}$.
- $\ell a^{n_0} b^{n_1} \vdash \ell' a^{n_0'} b^{n_1'}$ is checkable by DPDA.

Proof Idea

- Player I produces configurations c_0, c_1, ...
- Player 0 can check once whether $c_i \vdash c_{i+1}$ holds.

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Example

```
0:  INC(X0)
1:  INC(X1)
2:  IF(X1=0) GOTO 5
3:  DEC(X0)
...
```

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Example

```
$  0  $  1  a  $
```

```
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Example

$ 0 $ 1 a $
N

```
0:  INC(X0)
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N: Player O claims no error.

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Example

$ 0 $ 1 a $ 2 a b $ 3 a
N

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Example

$ 0 $ 1 a $ 2 a b $ 3 a b $ 4 a b $
N -

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N	-	N													

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N	-	N	-	-	N	-	-								

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Example

\$	0	\$	1	a	\$	2	a	b	\$	3	a	b	\$	4	a	b \$
N	-	N	-	-	N	-	-	-	R_0							

```
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3:  DEC(X0)
...
```

R_0: Player O claims error in `X0`.

Player O wins:
$(3, 1, 1) \not\vdash (4, 1, 1)$

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N	-	N	-	-	N	-	-	-	R_0							

- If machine halts, Player I has to cheat. Player O can detect this with linear delay and wins.
- If machine does not halt, Player I can play forever without cheating and wins.

More Undecidability

Corollary

The following problems are undecidable:

- **Input:** *Parity-DPDA $\mathcal{A}$.*
  Question: *Does Player O win the game induced by $L(\mathcal{A})$ with constant delay?*

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Undecidability results hold for visibly one-counter winning conditions: let Player O control the stack.

Outline

1. Definitions
2. Undecidability Results
3. **Lower Bounds on Delay**
4. Conclusion

Lower Bounds on Delay

Theorem

There exists a parity-DPDA $\mathcal{A}$ such that Player O wins the game induced by $L(\mathcal{A})$ with finite delay, but for any elementary delay function f, the game $\Gamma_f(L(\mathcal{A}))$ is won by Player I.

Lower Bounds on Delay

Theorem

There exists a parity-DPDA $\mathcal{A}$ such that Player O wins the game induced by $L(\mathcal{A})$ with finite delay, but for any elementary delay function f, the game $\Gamma_f(L(\mathcal{A}))$ is won by Player I.

Proof Idea

Preliminaries:

- Adapt idea from undecidability proof.
- Player I produces blocks on which a successor relation is defined (which can be checked by a DPDA).
- Player I has to cheat at some point.
- Player O wins if she catches Player I.

Proof Idea

- 0-th block: $w_0 = 0$.
- $(n+1)$-st block: $w_{n+1} = \$\,0\,\$\,00\,\$\,0000\,\$\;\cdots\;\$\,0^{2^{|w_n|}}\,\$$.
- $|w_{n+1}| > \sum_{i=0}^{|w_n|} 2^i = 2^{|w_n|+1} - 1$.

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Now, both players have to announce errors:

- Copy error: $|w_{n+1}|_\$ \neq |w_n| + 1$.
- Doubling error: infix $\$0^m\$0^n\$$ s.t. $n \neq 2m$.
- Both errors can be checked by a DPDA.

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Player O wins if:

- she claims an existing error before Player I does, or
- Player I claims an non-existing error, or
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Player O needs non-elementary lookahead to win this game.

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1. Definitions
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Conclusion

Delay games with context-free winning conditions.

- Determining the winner is undecidable.
- This holds even for restricted classes of winning conditions.

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Delay games with context-free winning conditions.

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- This holds even for restricted classes of winning conditions.
- Non-elementary lower bounds on delay.
- Again, also for restricted classes of winning conditions.

Open questions:

Undecidability and non-elementary lower bounds, if Player O controls the stack.

- What if Player I controls the stack?
- Linear delay necessary in this case.