The Complexity of Counting Models of Linear-time Temporal Logic

Joint work with Hazem Torfah (Saarland University)

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Research Training Group AlgoSyn
RWTH Aachen University

Why Model Counting

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How many models does a boolean formula φ have?

- Generalization of satisfiability: does φ have a model?
- Applications:
  - probabilistic inference problems
  - planning problems
  - combinatorial designs
  - etc.

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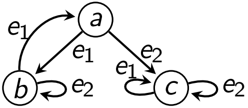
- .. count tree models of depth k with back-edges at leaves:
  - Analogue to synthesis: count the number of implementations (implementation freedom).



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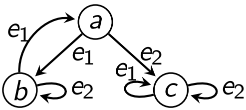

Theorem (Finkbeiner and Torfah '14)

1. *Word models can be counted in time* $\mathcal{O}(k \cdot 2^{2^{|\varphi|}})$.
2. *Tree models can be counted in time* $\mathcal{O}(k \cdot 2^{2^{2^{|\varphi|}}})$.

Outline

1. **Counting Complexity**
2. Counting Word Models
3. Counting Tree Models
4. Conclusion

Counting Complexity

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- Completeness: hardness and membership.

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Counting versions of easy problems can be hard!

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- $f \in \#2\textsc{Exptime}$ implies $f(w) \in \mathcal{O}(2^{2^{2^{p(|w|)}}})$ for a polynomial p.
- $w \mapsto 2^{2^{|w|}}$ is in $\#\textsc{Pspace}$.
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Counting Word-Models for Binary Bounds

Theorem

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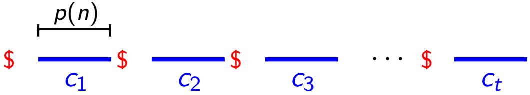

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- **Upper bound:** guess word of length k letter-by-letter (starting at the end) and model-check it on the fly (using unambiguous non-determinism). Then: one accepting run per model.

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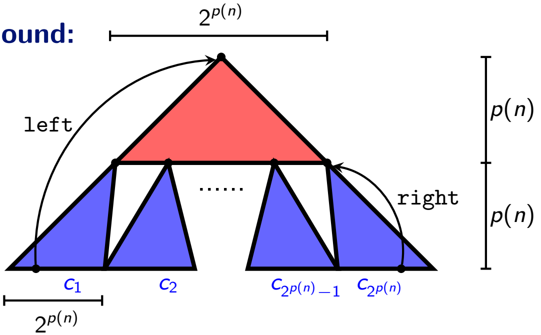

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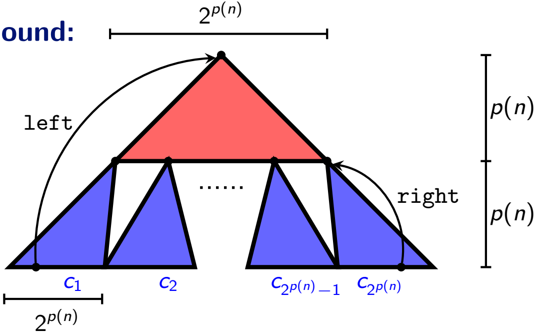

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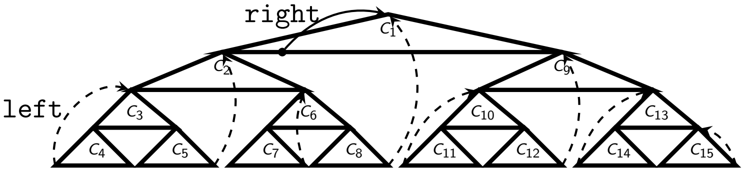

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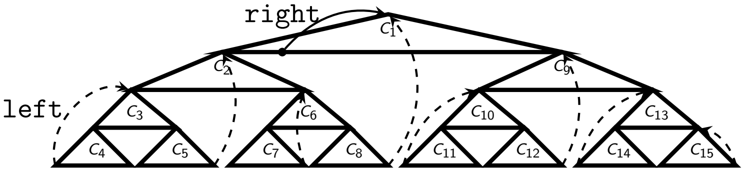

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- Close the gap for graph models, too.