Optimal Bounds in Parametric LTL Games

Martin Zimmermann

Universität des Saarlandes

October 28th, 2013

AVACS Meeting
Freiburg, Germany

Motivation

Linear Temporal Logic (LTL) as specification language:

- Simple and variable-free syntax and intuitive semantics.
- Expressively equivalent to first-order logic on words.
- LTL model-checking routinely applied in industrial settings.

But LTL cannot express timing constraints.

Motivation

Linear Temporal Logic (LTL) as specification language:

- Simple and variable-free syntax and intuitive semantics.
- Expressively equivalent to first-order logic on words.
- LTL model-checking routinely applied in industrial settings.

But LTL cannot express timing constraints. Possible remedies:

- Add $\mathbf{F}_{\leq k}$ for $k \in \mathbb{N}$. Problem: finding "right" k impracticable.

Motivation

Linear Temporal Logic (LTL) as specification language:

- Simple and variable-free syntax and intuitive semantics.
- Expressively equivalent to first-order logic on words.
- LTL model-checking routinely applied in industrial settings.

But LTL cannot express timing constraints. Possible remedies:

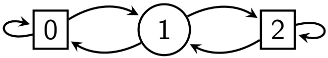
- Add $\mathbf{F}_{\leq k}$ for $k \in \mathbb{N}$. Problem: finding "right" k impracticable.
- Alur et. al, Kupferman et. al: add $\mathbf{F}_{\leq x}$ for variable x. Now:
  - does there exist a value x such that $\mathbf{F}_{\leq x}\varphi$ holds?
  - what is the best value x such that $\mathbf{F}_{\leq x}\varphi$ holds?

In Model-Checking: adding variable time bounds does not increase complexity.

Infinite Games

Arena $\mathcal{A} = (V, V_0, V_1, E)$:

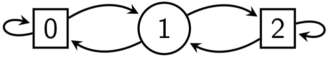
- finite directed graph (V, E),
- $V_0 \subseteq V$ positions of Player 0 (circles),
- $V_1 = V \setminus V_0$ positions of Player 1 (squares).



Infinite Games

Arena $\mathcal{A} = (V, V_0, V_1, E)$:

- finite directed graph (V, E),
- $V_0 \subseteq V$ positions of Player 0 (circles),
- $V_1 = V \setminus V_0$ positions of Player 1 (squares).



- Play: path $\rho_0 \rho_1 \cdots$ through $\mathcal{A}$.
- Strategy for Player i: $\sigma \colon V^* V_i \to V$ s.t. $(v, \sigma(wv)) \in E$.
- $\rho_0 \rho_1 \cdots$ consistent with σ: $\rho_{n+1} = \sigma(\rho_0 \cdots \rho_n)$ for all n s.t. $\rho_n \in V_i$.
- Finite-state strategy: implemented by finite automaton with output.

PLTL: Syntax and Semantics

Parametric LTL: p atomic proposition, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ ($\mathcal{X} \cap \mathcal{Y} = \emptyset$).

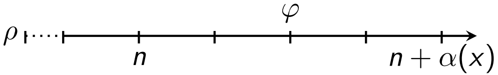
- $\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \varphi\mathbf{U}\varphi \mid \varphi\mathbf{R}\varphi \mid \mathbf{F}_{\leq x}\varphi \mid \mathbf{G}_{\leq y}\varphi$

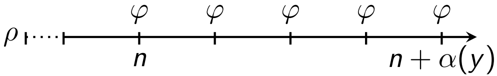
PLTL: Syntax and Semantics

Parametric LTL: p atomic proposition, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ ($\mathcal{X} \cap \mathcal{Y} = \emptyset$).

- $\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \varphi\mathbf{U}\varphi \mid \varphi\mathbf{R}\varphi \mid \mathbf{F}_{\leq x}\varphi \mid \mathbf{G}_{\leq y}\varphi$

Semantics w.r.t. variable valuation $\alpha\colon \mathcal{X} \cup \mathcal{Y} \to \mathbb{N}$:

- As usual for LTL operators.
- $(\rho, n, \alpha) \models \mathbf{F}_{\leq x}\varphi$: 

- $(\rho, n, \alpha) \models \mathbf{G}_{\leq y}\varphi$: 

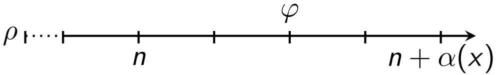
PLTL: Syntax and Semantics

Parametric LTL: p atomic proposition, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ ($\mathcal{X} \cap \mathcal{Y} = \emptyset$).

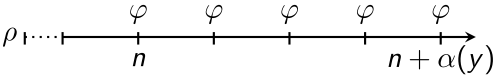
- $\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U} \varphi \mid \varphi \mathbf{R} \varphi \mid \mathbf{F}_{\leq x}\varphi \mid \mathbf{G}_{\leq y}\varphi$

Semantics w.r.t. variable valuation $\alpha \colon \mathcal{X} \cup \mathcal{Y} \to \mathbb{N}$:

- As usual for LTL operators.
- $(\rho, n, \alpha) \models \mathbf{F}_{\leq x}\varphi$:



- $(\rho, n, \alpha) \models \mathbf{G}_{\leq y}\varphi$:



Fragments:

- $\mathrm{PLTL}_{\mathbf{F}}$: no parameterized always operators $\mathbf{G}_{\leq y}$.
- $\mathrm{PLTL}_{\mathbf{G}}$: no parameterized eventually operators $\mathbf{F}_{\leq x}$.

PLTL Games

PLTL game: $\mathcal{G} = (\mathcal{A}, v_0, \varphi)$ with arena $\mathcal{A}$ (labeled by $\ell \colon V \to 2^P$), initial vertex v_0, and PLTL formula φ.

PLTL Games

PLTL game: $\mathcal{G} = (\mathcal{A}, v_0, \varphi)$ with arena $\mathcal{A}$ (labeled by $\ell \colon V \to 2^P$), initial vertex v_0, and PLTL formula φ.

Rules:

- all plays start in v_0.
- Player 0 wins $\rho_0\rho_1\cdots$ w.r.t. α, if $(\ell(\rho_0)\ell(\rho_1)\cdots, \alpha) \models \varphi$.
- Player 1 wins $\rho_0\rho_1\cdots$ w.r.t. α, if $(\ell(\rho_0)\ell(\rho_1)\cdots, \alpha) \not\models \varphi$.

PLTL Games

PLTL game: $\mathcal{G} = (\mathcal{A}, v_0, \varphi)$ with arena $\mathcal{A}$ (labeled by $\ell \colon V \to 2^P$), initial vertex v_0, and PLTL formula φ.

Rules:

- all plays start in v_0.
- Player 0 wins $\rho_0\rho_1\cdots$ w.r.t. α, if $(\ell(\rho_0)\ell(\rho_1)\cdots, \alpha) \models \varphi$.
- Player 1 wins $\rho_0\rho_1\cdots$ w.r.t. α, if $(\ell(\rho_0)\ell(\rho_1)\cdots, \alpha) \not\models \varphi$.
- σ is winning strategy for Player i w.r.t. α, if every consistent play is winning for Player i w.r.t. α.
- Winning valuations for Player i

$$\mathcal{W}_i(\mathcal{G}) = \{\alpha \mid \text{Player } i \text{ has winning strategy for } \mathcal{G} \text{ w.r.t. } \alpha\}$$

PLTL Games

PLTL game: $\mathcal{G} = (\mathcal{A}, v_0, \varphi)$ with arena $\mathcal{A}$ (labeled by $\ell \colon V \to 2^P$), initial vertex v_0, and PLTL formula φ.

Rules:

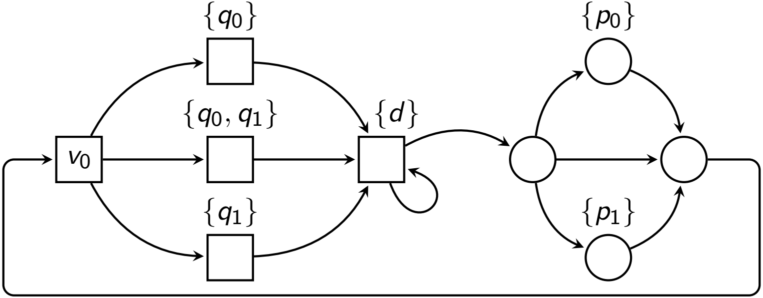
- all plays start in v_0.
- Player 0 wins $\rho_0\rho_1\cdots$ w.r.t. α, if $(\ell(\rho_0)\ell(\rho_1)\cdots, \alpha) \models \varphi$.
- Player 1 wins $\rho_0\rho_1\cdots$ w.r.t. α, if $(\ell(\rho_0)\ell(\rho_1)\cdots, \alpha) \not\models \varphi$.
- σ is winning strategy for Player i w.r.t. α, if every consistent play is winning for Player i w.r.t. α.
- Winning valuations for Player i

$$\mathcal{W}_i(\mathcal{G}) = \{\alpha \mid \text{Player } i \text{ has winning strategy for } \mathcal{G} \text{ w.r.t. } \alpha\}$$

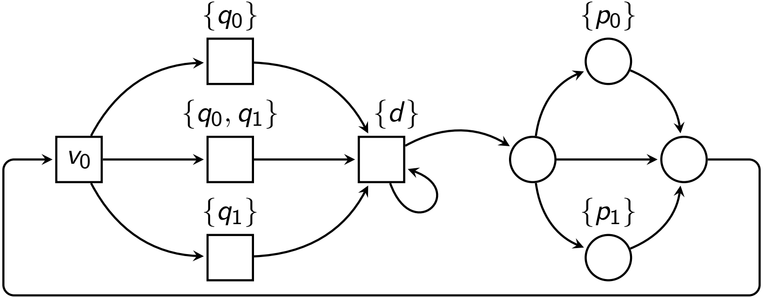
Lemma

Determinacy: $\mathcal{W}_0(\mathcal{G})$ is the complement of $\mathcal{W}_1(\mathcal{G})$.

An Example

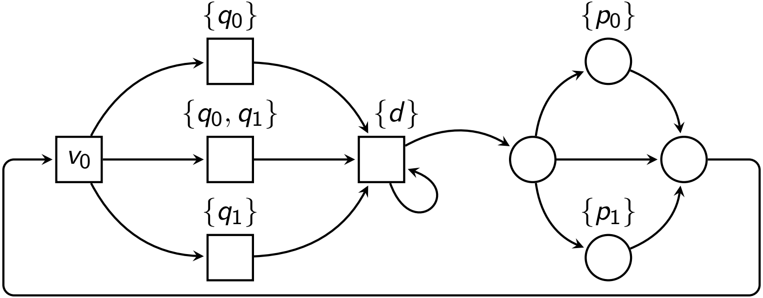


An Example



- $\varphi_1 = \mathbf{FG}d \vee \bigwedge_{i \in \{0,1\}} \mathbf{G}(q_i \rightarrow \mathbf{F}p_i) \;:\; \mathcal{W}_1(\mathcal{G}_1) = \emptyset$.

An Example



- $\varphi_1 = \mathbf{FG}d \vee \bigwedge_{i\in\{0,1\}} \mathbf{G}(q_i \rightarrow \mathbf{F}p_i)\ :\ \mathcal{W}_1(\mathcal{G}_1) = \emptyset.$
- $\varphi_2 = \mathbf{FG}d \vee \bigwedge_{i\in\{0,1\}} \mathbf{G}(q_i \rightarrow \mathbf{F}_{\leq x_i}p_i)\ :\ \mathcal{W}_0(\mathcal{G}_2) = \emptyset.$

More Example Properties

- Bounded Büchi: $\mathbf{GF}_{\leq x} p$

More Example Properties

- Bounded Büchi: $\mathbf{GF}_{\leq x} p$
- Finitary parity (Chatterjee, Henzinger, Horn):

$$\mathbf{FG} \bigwedge_{c \text{ odd}} \left(c \rightarrow \mathbf{F}_{\leq x} \bigvee_{\substack{c' > c \\ c' \text{ even}}} c' \right)$$

More Example Properties

- Bounded Büchi: $\mathbf{GF}_{\leq x} p$
- Finitary parity (Chatterjee, Henzinger, Horn):

$$\mathbf{FG} \bigwedge_{c \text{ odd}} \left(c \rightarrow \mathbf{F}_{\leq x} \bigvee_{\substack{c' > c \\ c' \text{ even}}} c' \right)$$

- Finitary Streett (CHH):

$$\mathbf{FG} \bigwedge_{j=1}^{k} (R_j \rightarrow \mathbf{F}_{\leq x} G_j)$$

Decision Problems

- Membership: given $\mathcal{G}$, $i \in \{0, 1\}$, and α, is $\alpha \in \mathcal{W}_i(\mathcal{G})$?
- Emptiness: given $\mathcal{G}$ and $i \in \{0, 1\}$, is $\mathcal{W}_i(\mathcal{G})$ empty?
- Finiteness: given $\mathcal{G}$ and $i \in \{0, 1\}$, is $\mathcal{W}_i(\mathcal{G})$ finite?
- Universality: given $\mathcal{G}$ and $i \in \{0, 1\}$, is $\mathcal{W}_i(\mathcal{G})$ universal?

Decision Problems

- Membership: given $\mathcal{G}$, $i \in \{0,1\}$, and α, is $\alpha \in \mathcal{W}_i(\mathcal{G})$?
- Emptiness: given $\mathcal{G}$ and $i \in \{0,1\}$, is $\mathcal{W}_i(\mathcal{G})$ empty?
- Finiteness: given $\mathcal{G}$ and $i \in \{0,1\}$, is $\mathcal{W}_i(\mathcal{G})$ finite?
- Universality: given $\mathcal{G}$ and $i \in \{0,1\}$, is $\mathcal{W}_i(\mathcal{G})$ universal?

The benchmark:

Theorem (Pnueli, Rosner 1989)

Solving LTL *games is* **2Exptime***-complete.*

Decision Problems

- Membership: given $\mathcal{G}$, $i \in \{0,1\}$, and α, is $\alpha \in \mathcal{W}_i(\mathcal{G})$?
- Emptiness: given $\mathcal{G}$ and $i \in \{0,1\}$, is $\mathcal{W}_i(\mathcal{G})$ empty?
- Finiteness: given $\mathcal{G}$ and $i \in \{0,1\}$, is $\mathcal{W}_i(\mathcal{G})$ finite?
- Universality: given $\mathcal{G}$ and $i \in \{0,1\}$, is $\mathcal{W}_i(\mathcal{G})$ universal?

The benchmark:

Theorem (Pnueli, Rosner 1989)

Solving LTL *games is* **2Exptime***-complete.*

Adding parameterized operators does not increase complexity:

Theorem

All four decision problems are **2Exptime***-complete.*

Proof Idea

Emptiness for $\mathrm{PLTL}_{\mathbf{F}}$ games, i.e., only $\mathbf{F}_{\leq x}$ in φ.

1. Duplicate arena, color one copy red, the other green. Player 0 can change between copies after every move.
2. Inductively replace every $F_{\leq x}\psi$ by

$$(\mathit{red} \rightarrow (\mathit{red}\,\mathbf{U}(\mathit{green}\,\mathbf{U}\psi))) \wedge (\mathit{green} \rightarrow (\mathit{green}\,\mathbf{U}(\mathit{red}\,\mathbf{U}\psi)))$$

3. Add conjunct $\mathbf{GF}\mathit{red} \wedge \mathbf{GF}\mathit{green}$ to φ, obtain φ'.

Proof Idea

Emptiness for $\mathrm{PLTL}_{\mathbf{F}}$ games, i.e., only $\mathbf{F}_{\leq x}$ in φ.

1. Duplicate arena, color one copy red, the other green. Player 0 can change between copies after every move.
2. Inductively replace every $F_{\leq x}\psi$ by

$$(red \rightarrow (red\,\mathbf{U}(green\,\mathbf{U}\psi))) \wedge (green \rightarrow (green\,\mathbf{U}(red\,\mathbf{U}\psi)))$$

3. Add conjunct $\mathbf{GF}red \wedge \mathbf{GF}green$ to φ, obtain φ'.
4. Player 0 wins LTL game $(\mathcal{A}', \varphi')$ iff there exists α s.t. Player 0 wins $(\mathcal{A}, \varphi)$ w.r.t. α.
5. Proof relies on finite-state determinacy of LTL games.

Proof Idea

Emptiness for $\mathrm{PLTL}_{\mathbf{F}}$ games, i.e., only $\mathbf{F}_{\leq x}$ in φ.

1. Duplicate arena, color one copy red, the other green. Player 0 can change between copies after every move.
2. Inductively replace every $F_{\leq x}\psi$ by

$$(\mathit{red} \to (\mathit{red}\,\mathbf{U}(\mathit{green}\,\mathbf{U}\psi))) \wedge (\mathit{green} \to (\mathit{green}\,\mathbf{U}(\mathit{red}\,\mathbf{U}\psi)))$$

3. Add conjunct $\mathbf{GF}\mathit{red} \wedge \mathbf{GF}\mathit{green}$ to φ, obtain φ'.
4. Player 0 wins LTL game $(\mathcal{A}', \varphi')$ iff there exists α s.t. Player 0 wins $(\mathcal{A}, \varphi)$ w.r.t. α.
5. Proof relies on finite-state determinacy of LTL games.

Corollary: doubly-exponential upper bound on α.

Proof Idea

Emptiness for $\mathrm{PLTL}_{\mathbf{F}}$ games, i.e., only $\mathbf{F}_{\leq x}$ in φ.

1. Duplicate arena, color one copy red, the other green. Player 0 can change between copies after every move.
2. Inductively replace every $F_{\leq x}\psi$ by

$$(\mathit{red} \rightarrow (\mathit{red}\,\mathbf{U}(\mathit{green}\,\mathbf{U}\psi))) \wedge (\mathit{green} \rightarrow (\mathit{green}\,\mathbf{U}(\mathit{red}\,\mathbf{U}\psi)))$$

3. Add conjunct $\mathbf{GF}\mathit{red} \wedge \mathbf{GF}\mathit{green}$ to φ, obtain φ'.
4. Player 0 wins LTL game $(\mathcal{A}', \varphi')$ iff there exists α s.t. Player 0 wins $(\mathcal{A}, \varphi)$ w.r.t. α.
5. Proof relies on finite-state determinacy of LTL games.

Corollary: doubly-exponential upper bound on α.

Full PLTL and other problems: use monotonicity and duality of $\mathbf{F}_{\leq x}$ and $\mathbf{G}_{\leq y}$

Optimization Problems

For $\mathrm{PLTL}_{\mathbf{F}}$ and $\mathrm{PLTL}_{\mathbf{G}}$ winning conditions, synthesis is an optimization problem: what is the best variable valuation in $\mathcal{W}_i(\mathcal{G})$?

Optimization Problems

For $\mathrm{PLTL}_{\mathbf{F}}$ and $\mathrm{PLTL}_{\mathbf{G}}$ winning conditions, synthesis is an optimization problem: what is the best variable valuation in $\mathcal{W}_i(\mathcal{G})$?

Theorem

Let $\mathcal{G}_{\mathbf{F}}$ be a $\mathrm{PLTL}_{\mathbf{F}}$ game with winning condition $\varphi_{\mathbf{F}}$ and let $\mathcal{G}_{\mathbf{G}}$ be a $\mathrm{PLTL}_{\mathbf{G}}$ game with winning condition $\varphi_{\mathbf{G}}$. The following values (and winning strategies realizing them) can be computed in **triply**-*exponential time.*

1. $\min_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{F}})} \min_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x)$.

Optimization Problems

For $\mathrm{PLTL}_{\mathbf{F}}$ and $\mathrm{PLTL}_{\mathbf{G}}$ winning conditions, synthesis is an optimization problem: what is the best variable valuation in $\mathcal{W}_i(\mathcal{G})$?

Theorem

Let $\mathcal{G}_{\mathbf{F}}$ be a $\mathrm{PLTL}_{\mathbf{F}}$ game with winning condition $\varphi_{\mathbf{F}}$ and let $\mathcal{G}_{\mathbf{G}}$ be a $\mathrm{PLTL}_{\mathbf{G}}$ game with winning condition $\varphi_{\mathbf{G}}$. The following values (and winning strategies realizing them) can be computed in **triply**-*exponential time.*

1. $\min_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{F}})} \min_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x)$.
2. $\min_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{F}})} \max_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x)$.

Optimization Problems

For $\mathrm{PLTL}_{\mathbf{F}}$ and $\mathrm{PLTL}_{\mathbf{G}}$ winning conditions, synthesis is an optimization problem: what is the best variable valuation in $\mathcal{W}_i(\mathcal{G})$?

Theorem

Let $\mathcal{G}_{\mathbf{F}}$ be a $\mathrm{PLTL}_{\mathbf{F}}$ game with winning condition $\varphi_{\mathbf{F}}$ and let $\mathcal{G}_{\mathbf{G}}$ be a $\mathrm{PLTL}_{\mathbf{G}}$ game with winning condition $\varphi_{\mathbf{G}}$. The following values (and winning strategies realizing them) can be computed in **triply**-*exponential time.*

1. $\min_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{F}})} \min_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x)$.
2. $\min_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{F}})} \max_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x)$.
3. $\max_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{G}})} \max_{y \in \mathrm{var}(\varphi_{\mathbf{G}})} \alpha(y)$.
4. $\max_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{G}})} \min_{y \in \mathrm{var}(\varphi_{\mathbf{G}})} \alpha(y)$.

All values are at most doubly-exponential in the size of the game.

Proof Idea

1. All problems reducible to $\min_{\alpha \in \mathcal{W}_0(\mathcal{G})} \alpha(x)$ for φ with $\mathrm{var}(\varphi) = \{x\}$.
2. Recall: algorithm for emptiness of $\mathcal{W}_0(\mathcal{G})$ yields doubly-exponential upper bound b on $\min_{\alpha \in \mathcal{W}_0(\mathcal{G})} \alpha(x)$.
3. For every $n \in [0, b]$ test whether n is optimum:

Proof Idea

1. All problems reducible to $\min_{\alpha \in \mathcal{W}_0(\mathcal{G})} \alpha(x)$ for φ with $\mathrm{var}(\varphi) = \{x\}$.
2. Recall: algorithm for emptiness of $\mathcal{W}_0(\mathcal{G})$ yields doubly-exponential upper bound b on $\min_{\alpha \in \mathcal{W}_0(\mathcal{G})} \alpha(x)$.
3. For every $n \in [0, b]$ test whether n is optimum:
   - **3.1** Translate φ into Büchi automaton $\mathfrak{A}_\varphi$ (treat $\mathbf{F}_{\leq x}$ as $\mathbf{F}$).
   - **3.2** Add a counter with range $[0, n]$ for every occurence of x to simulate semantics of $\mathbf{F}_{\leq x}$, obtain $\mathfrak{A}'_\varphi$ of size $2^{|\varphi|} \cdot n^{|\varphi|}$.
   - **3.3** Determize $\mathfrak{A}'_\varphi$ to obtain parity automaton $\mathfrak{P}_\varphi$ of size $2^{\mathcal{O}(|\varphi|^2 \cdot (2n)^{2|\varphi|})}$ and $\mathcal{O}(|\varphi| \cdot n^{|\varphi|})$ colors.
   - **3.4** Solve the parity game $\mathcal{A} \times \mathfrak{P}_\varphi$.

Proof Idea

1. All problems reducible to $\min_{\alpha \in \mathcal{W}_0(\mathcal{G})} \alpha(x)$ for φ with $\mathrm{var}(\varphi) = \{x\}$.
2. Recall: algorithm for emptiness of $\mathcal{W}_0(\mathcal{G})$ yields doubly-exponential upper bound b on $\min_{\alpha \in \mathcal{W}_0(\mathcal{G})} \alpha(x)$.
3. For every $n \in [0, b]$ test whether n is optimum:
   - **3.1** Translate φ into Büchi automaton $\mathfrak{A}_\varphi$ (treat $\mathbf{F}_{\leq x}$ as $\mathbf{F}$).
   - **3.2** Add a counter with range $[0, n]$ for every occurence of x to simulate semantics of $\mathbf{F}_{\leq x}$, obtain $\mathfrak{A}'_\varphi$ of size $2^{|\varphi|} \cdot n^{|\varphi|}$.
   - **3.3** Determize $\mathfrak{A}'_\varphi$ to obtain parity automaton $\mathfrak{P}_\varphi$ of size $2^{\mathcal{O}(|\varphi|^2 \cdot (2n)^{2|\varphi|})}$ and $\mathcal{O}(|\varphi| \cdot n^{|\varphi|})$ colors.
   - **3.4** Solve the parity game $\mathcal{A} \times \mathfrak{P}_\varphi$.

Algorithm has triply exponential running time, since n is at most doubly-exponential.

Lower Bounds

For $\mathrm{PLTL}_{\mathbf{F}}$ games: doubly-exponential lower bound

Theorem

For every $n \geq 1$, there exists a $\mathrm{PLTL}_{\mathbf{F}}$ game $\mathcal{G}_n$ with winning condition φ_n with $|\mathcal{G}_n| \in \mathcal{O}(n^2)$ and $\mathrm{var}(\varphi_n) = \{x\}$ such that $\mathcal{W}_0(\mathcal{G}_n) \neq \emptyset$, but Player 1 wins $\mathcal{G}_n$ with respect to every variable valuation α such that $\alpha(x) \leq 2^{2^n}$.

Lower Bounds

For $\mathrm{PLTL}_{\mathsf{F}}$ games: doubly-exponential lower bound

Theorem

For every $n \geq 1$, there exists a $\mathrm{PLTL}_{\mathsf{F}}$ game $\mathcal{G}_n$ with winning condition φ_n with $|\mathcal{G}_n| \in \mathcal{O}(n^2)$ and $\mathrm{var}(\varphi_n) = \{x\}$ such that $\mathcal{W}_0(\mathcal{G}_n) \neq \emptyset$, but Player 1 wins $\mathcal{G}_n$ with respect to every variable valuation α such that $\alpha(x) \leq 2^{2^n}$.

For $\mathrm{PLTL}_{\mathsf{G}}$ games: doubly-exponential lower bound (by duality)

Theorem

For every $n \geq 1$, there exists a $\mathrm{PLTL}_{\mathsf{G}}$ game $\mathcal{G}_n$ with winning condition φ_n with $|\mathcal{G}_n| \in \mathcal{O}(n^2)$ and $\mathrm{var}(\varphi_n) = \{y\}$ such that $\mathcal{W}_0(\mathcal{G}_n)$ is not universal, but Player 0 wins $\mathcal{G}_n$ with respect to every variable valuation α such that $\alpha(x) \leq 2^{2^n}$.

Open Problems

- Optimization problems in **2Exptime**?

Open Problems

- Optimization problems in **2Exptime**?
- Tradeoff quality vs. size of finite-state strategies.

Open Problems

- Optimization problems in **2Exptime**?
- Tradeoff quality vs. size of finite-state strategies.
- Change order of quantifiers: emptiness $\equiv \exists\sigma\exists\alpha\forall\rho$. What about $\exists\sigma\forall\rho\exists\alpha$ (non-uniform bounds)?

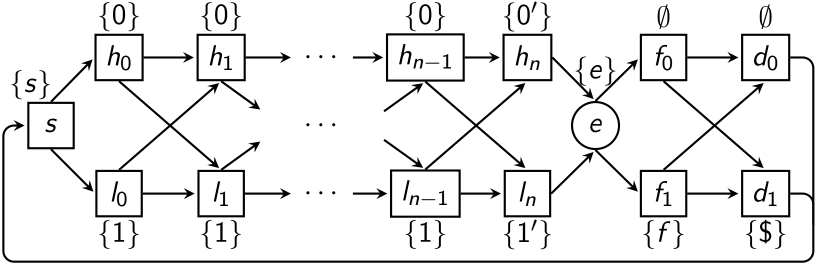
Open Problems

- Optimization problems in **2Exptime**?
- Tradeoff quality vs. size of finite-state strategies.
- Change order of quantifiers: emptiness $\equiv \exists\sigma\exists\alpha\forall\rho$. What about $\exists\sigma\forall\rho\exists\alpha$ (non-uniform bounds)?
- Explicit representation of $\mathcal{W}_i(\mathcal{G})$ for $\mathrm{PLTL}_{\mathbf{F}}$ and $\mathrm{PLTL}_{\mathbf{G}}$ games (upwards-closed and semi-linear)?
- How big has such a representation to be?

Open Problems

- Optimization problems in **2Exptime**?
- Tradeoff quality vs. size of finite-state strategies.
- Change order of quantifiers: emptiness $\equiv \exists\sigma\exists\alpha\forall\rho$. What about $\exists\sigma\forall\rho\exists\alpha$ (non-uniform bounds)?
- Explicit representation of $\mathcal{W}_i(\mathcal{G})$ for $\mathrm{PLTL}_{\mathbf{F}}$ and $\mathrm{PLTL}_{\mathbf{G}}$ games (upwards-closed and semi-linear)?
- How big has such a representation to be?
- Once again: Optimization problems in **2Exptime**?

The Game for the Lower Bounds



A Play in $\mathcal{G}_n$

We start in d_1. The trace of a play looks as follows:

$$\begin{aligned}\{\$\}\{s\}\{b_0^0\}\cdots\{b_{n-1}^0\}\{b_n^0\}\{e\}F_0 D_0 \\ \{s\}\{b_0^1\}\cdots\{b_{n-1}^1\}\{b_n^1\}\{e\}F_1 D_1 \\ \{s\}\{b_0^2\}\cdots\{b_{n-1}^2\}\{b_n^2\}\{e\}F_2 D_2 \cdots\end{aligned}$$

where

- $b_0^j, \ldots, b_{n-1}^j \in \{0,1\} \Rightarrow$ encoding of $c_j \in \{0,1,\ldots,2^n-1\}$
- $b_n^j \in \{0', 1'\}$
- F_j is $\{f\}$ or $\emptyset$ (a flag for Player 0)
- D_j is $\{\$\}$ or $\emptyset$ (a flag for Player 1)

A Play in $\mathcal{G}_n$

We start in d_1. The trace of a play looks as follows:

$$\begin{aligned}&\{\$\}\{s\}\{b_0^0\}\cdots\{b_{n-1}^0\}\{b_n^0\}\{e\}F_0D_0\\&\quad\{s\}\{b_0^1\}\cdots\{b_{n-1}^1\}\{b_n^1\}\{e\}F_1D_1\\&\quad\{s\}\{b_0^2\}\cdots\{b_{n-1}^2\}\{b_n^2\}\{e\}F_2D_2\cdots\end{aligned}$$

where

- $b_0^j,\dots,b_{n-1}^j \in \{0,1\} \Rightarrow$ encoding of $c_j \in \{0,1,\dots,2^n-1\}$
- $b_n^j \in \{0',1'\}$
- F_j is $\{f\}$ or $\emptyset$ (a flag for Player 0)
- D_j is $\{\$\}$ or $\emptyset$ (a flag for Player 1)

Infinitely many \$: primed bits encode numbers $d_\ell \in \mathbb{N}$

The Winning Condition

Recall: numbers c_j (adresses) and numbers d_ℓ whose bits are adressed by the c_j

There is an LTL formula ψ_1 which expresses:

1. Structure: Infinitely many \$
2. Initialization: after each \$, the next c_j is zero.
3. Increment: if $c_j < 2^n - 1$, then $c_{j+1} = c_j + 1$.
4. Reset: if c_j is $2^n - 1$, then it is followed by \$.

ψ_1 uses n nested next-operators to check condition 3.

The Winning Condition

Recall: numbers c_j (adresses) and numbers d_ℓ whose bits are adressed by the c_j

There is an LTL formula ψ_1 which expresses:

1. Structure: Infinitely many \$
2. Initialization: after each \$, the next c_j is zero.
3. Increment: if $c_j < 2^n - 1$, then $c_{j+1} = c_j + 1$.
4. Reset: if c_j is $2^n - 1$, then it is followed by \$.

ψ_1 uses n nested next-operators to check condition 3.

Lemma

$\varphi_1 \Rightarrow d_\ell \in \{0, 1, \dots, 2^{2^n} - 1\}$.

The Winning Condition, Part 2

$$\varphi_n = \psi_1 \rightarrow (\psi_f \wedge \psi_{\mathrm{err}} \wedge \mathbf{F}_{\leq x} f)$$

where

- ψ_f: exactly one f
- ψ_{err}: Player 0 used f to mark
  - a single bit that is incorrectly updated from d_ℓ to $d_{\ell+1}$ (formula uses adresses to verify this), or
  - a d_ℓ with $d_\ell = 2^{2^n} - 1$ (no primed 0 between two \$).

The Winning Condition, Part 2

$$\varphi_n = \psi_1 \rightarrow (\psi_f \wedge \psi_{\mathrm{err}} \wedge \mathbf{F}_{\leq x} f)$$

where

- ψ_f: exactly one f
- ψ_{err}: Player 0 used f to mark
  - a single bit that is incorrectly updated from d_ℓ to $d_{\ell+1}$ (formula uses adresses to verify this), or
  - a d_ℓ with $d_\ell = 2^{2^n} - 1$ (no primed 0 between two \$).

Player 0 wins, since Player 1 has to reach $2^{2^n} - 1$ or has to introduce an increment-error. But this can take more than $2^{2^n} - 1$ moves using correct updates.