Time-optimal Strategies for Infinite Games

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Introduction

Model Checking: program P, specification φ, does

$$P \models \varphi \ ?$$

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Synthesis: environment E, specification φ. Generate program P such that

$$E \times P \models \varphi \ .$$

Synthesis as a game: no matter what the environment does, the program has to guarantee φ.

- Beautiful and rich theory based on infinite graph games.
- typically: a player either wins or loses (zero-sum).
- here: adding quantitative aspects to infinite games.

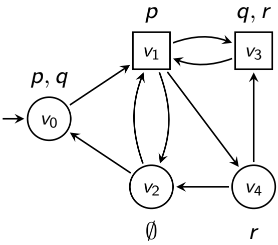
Outline

1. **Infinite Games**
2. Poset Games
3. Parametric LTL Games
4. Finite-time Muller Games
5. Conclusion

Definitions

An arena $\mathcal{A} = (V, V_0, V_1, E, v_0, l)$ consists of

- a finite directed graph (V, E) without dead-ends,
- a partition $\{V_0, V_1\}$ of V denoting the positions of Player 0 (circles) and Player 1 (squares),
- an initial vertex $v_0 \in V$,
- a labeling function $l : V \to 2^P$ for some set P of atomic propositions.



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- Strategy for Player $i \in \{0,1\}$: mapping $\sigma : V^*V_i \rightarrow V$ such that $(s, \sigma(ws)) \in E$.
- σ is finite-state: σ computable by finite automaton with output.
- $\rho_0\rho_1\rho_2\ldots$ is consistent with σ: $\rho_{n+1} = \sigma(\rho_0 \ldots \rho_n)$ for all n such that $\rho_n \in V_i$.

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Game: $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ with $\mathrm{Win} \subseteq V^\omega$.

- ρ winning for Player 0: $\rho \in \mathrm{Win}$.
- ρ winning for Player 1: $\rho \in V^\omega \setminus \mathrm{Win}$.

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- ρ winning for Player 0: $\rho \in \mathrm{Win}$.
- ρ winning for Player 1: $\rho \in V^\omega \setminus \mathrm{Win}$.
- σ winning strategy for Player i: all plays ρ consistent with σ are winning for Player i.
- $\mathcal{G}$ determined: one player has a winning strategy.

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Motivation

- Request-Reponse conditions are a typical requirement on reactive systems.
- There is a natural definition of waiting times and they allow time-optimal strategies.

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- There is a natural definition of waiting times and they allow time-optimal strategies.

Goal:

- Extend the Request-Response condition to partially ordered objectives..
- .. while retaining the notion of waiting times and the existence of time-optimal strategies.

Request-Response games

Request-response game: $(\mathcal{A}, (Q_j, P_j)_{j=1,\ldots,k})$ where $Q_j, P_j \subseteq V$. Player 0 wins a play if every visit to Q_j (request) is responded by a later visit to P_j.

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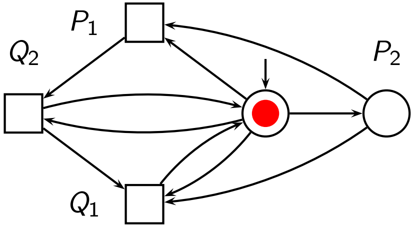

$t_1 : \quad 0$

$t_2 : \quad 0$

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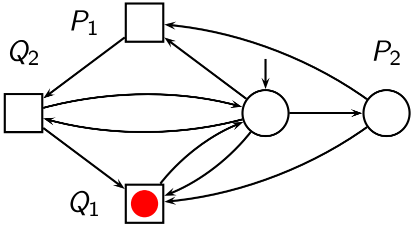

t_1 : 0 1
t_2 : 0 0

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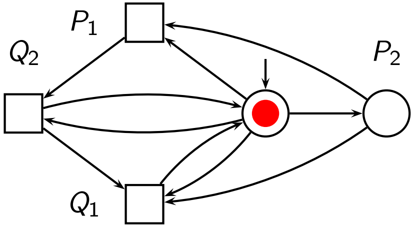

t_1 : 0 1 2

t_2 : 0 0 0

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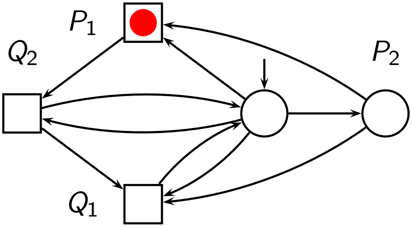

$t_1: \quad 0 \quad 1 \quad 2 \quad 0$

$t_2: \quad 0 \quad 0 \quad 0 \quad 0$

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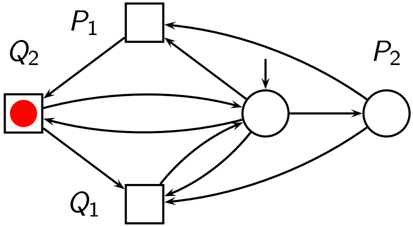

$t_1:\quad 0\quad 1\quad 2\quad 0\quad 0$

$t_2:\quad 0\quad 0\quad 0\quad 0\quad 1$

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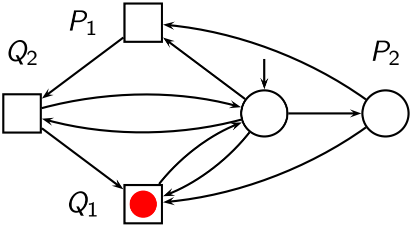

t_1 :	0	1	2	0	0	1
t_2 :	0	0	0	0	1	2

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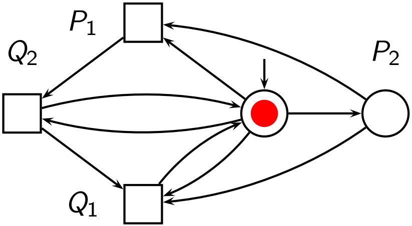

t_1 :	0	1	2	0	0	1	2
t_2 :	0	0	0	0	1	2	3

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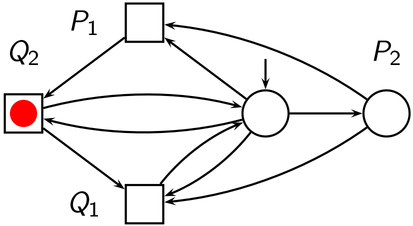

t_1 :	0	1	2	0	0	1	2	3
t_2 :	0	0	0	0	1	2	3	4

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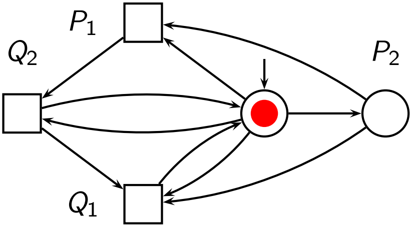

t_1 :	0	1	2	0	0	1	2	3	4
t_2 :	0	0	0	0	1	2	3	4	5

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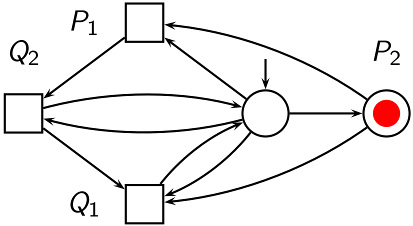

t_1 :	0	1	2	0	0	1	2	3	4	5
t_2 :	0	0	0	0	1	2	3	4	5	0

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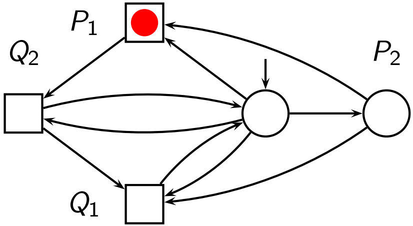

t_1 :	0	1	2	0	0	1	2	3	4	5	0
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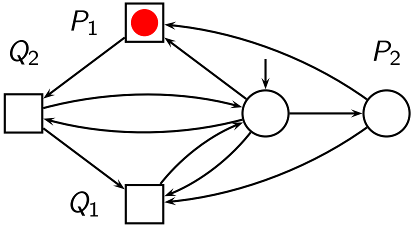

t_1 :	0	1	2	0	0	1	2	3	4	5	0
t_2 :	0	0	0	0	1	2	3	4	5	0	0
$p_i = t_1 + t_2$:	0	1	2	0	1	3	5	7	9	5	0

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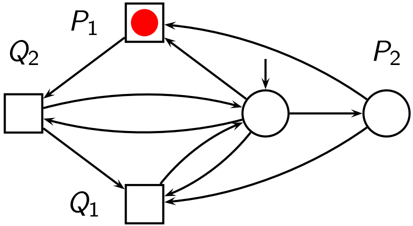

t_1 :	0	1	2	0	0	1	2	3	4	5	0
t_2 :	0	0	0	0	1	2	3	4	5	0	0
$p_i = t_1 + t_2$:	0	1	2	0	1	3	5	7	9	5	0
$\frac{1}{n}\sum_{i=1}^{n} p_i$:	0	$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{7}{6}$	$\frac{12}{7}$	$\frac{19}{8}$	$\frac{28}{9}$	$\frac{34}{10}$	$\frac{34}{11}$

Request-Reponse Games: Results

- Waiting times: start a clock for every request that is stopped as soon as it is responded (and ignore subsequent requests).
- Accumulated waiting time: sum up the clock values of a play prefix (quadratic influence of open requests).
- Value of a play: limit superior of the average accumulated waiting time.
- Value of a strategy: value of the worst play consistent with the strategy.

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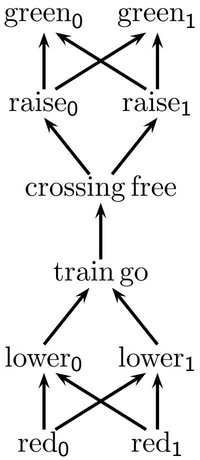
Theorem (Horn, Thomas, Wallmeier)

If Player 0 has a winning strategy for an RR-game, then she also has an optimal winning strategy, which is finite-state and effectively computable.

Extending Request-Reponse Games



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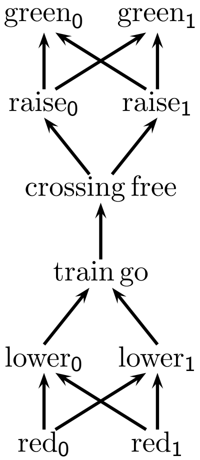
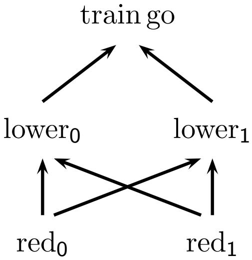


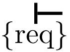
Generalize RR-games to express more complicated conditions, but retain notion of time-optimality.

Request: still a singular event.

Response: partially ordered set of events.

A Play





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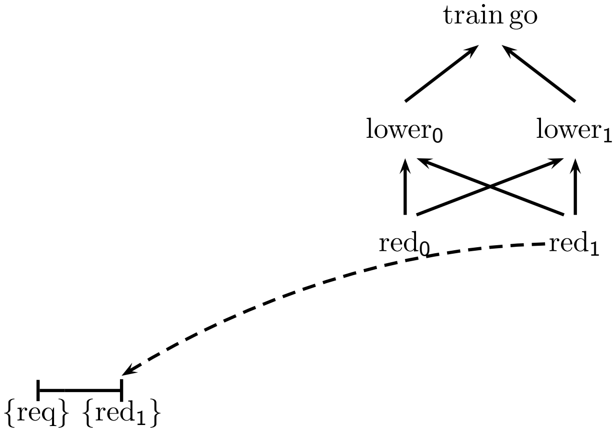
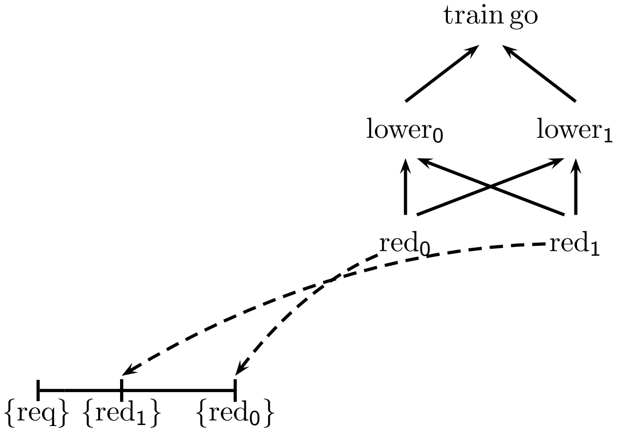
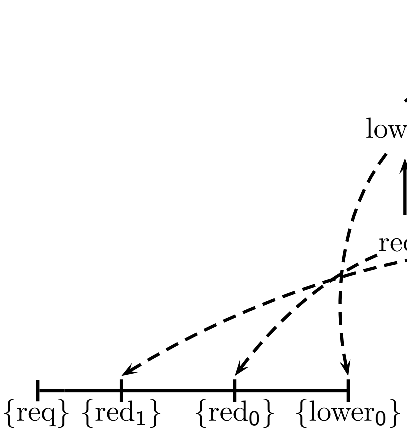


A Play

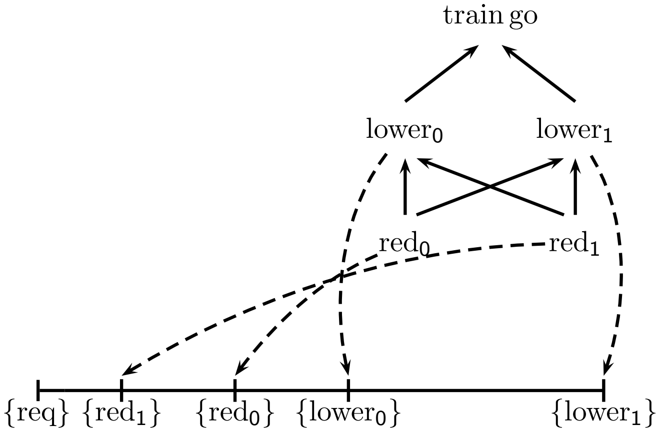


train go
lower_0
lower_1
red_0
red_1
{req}
$\{\text{red}_1\}$
$\{\text{red}_0\}$

A Play



A Play


train go
$lower_0$
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red_0
red_1
$\{req\}$
$\{red_1\}$
$\{red_0\}$
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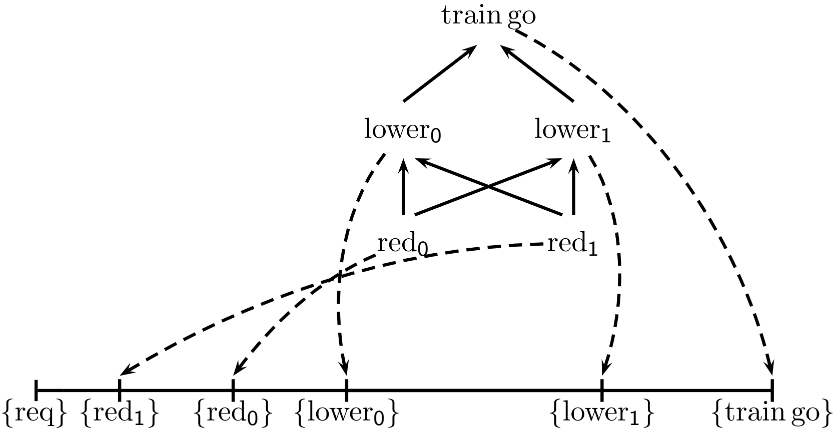
A Play



Winning condition for Player 0: every request q_j is responded by a later embedding of $\mathcal{P}_j$.

Solving Poset Games

Theorem

Poset games are determined with finite-state strategies, i.e., in every poset games, one of the players has a finite-state winning strategy.

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Proof:
Reduction to Büchi games; memory is used

- to store elements of the posets that still have to be embedded,
- to deal with overlapping embeddings,
- to implement a cyclic counter to ensure that every request is responded by an embedding.

Size of the memory: exponential in the size of the posets $\mathcal{P}_j$.

Waiting Times

As desired, a natural definition of waiting times is retained:

- Start a clock if a request is encountered...
- ... that is stopped as soon as the embedding is completed.
- Need a clock for every request (even if another request is already open).

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- Value of a play: limit superior of the average accumulated waiting time.
- Value of a strategy: value of the worst play consistent with the strategy.
- Corresponding notion of optimal strategies.

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Proof:

- If Player 0 has a winning strategy, then she also has one of value less than a certain constant c (from reduction). This bounds the value of an optimal strategy, too.
- For every strategy of value $\leq c$ there is another strategy of smaller or equal value, that also bounds all waiting times and bounds the number of open requests.
- If the waiting times and the number of open requests are bounded, then $\mathcal{G}$ can be reduced to a mean-payoff game.

Further research and Open Problems

Size of the mean-payoff game: super-exponential in the size of the poset game (holds already for RR-games). Needed: tight bounds on the length of a non-self-covering sequence of waiting time vectors.

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Also:

- Heuristic algorithms and approximatively optimal strategies.
- Lower bounds on the memory size of an optimal strategy.
- Direct computation of optimal strategies (without reduction to mean-payoff games).
- Other valuation functions for plays (e.g., discounting, $\limsup \sum_{i=1}^{k} t_i$).
- Tradeoff between size and value of a strategy.

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Motivation

Here, we consider winning conditions in linear temporal logic (LTL). Advantages of LTL as specification language are

- compact, variable-free syntax,
- intuitive semantics,
- successfully employed in model checking tools.

Drawback: LTL lacks capabilities to express timing constraints.

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Drawback: LTL lacks capabilities to express timing constraints.

Solution: Consider games with winning conditions in extensions of LTL that can express timing constraints.

LTL

Formulae of Linear temporal logic over P:

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \mathbf{F}\varphi \mid \mathbf{G}\varphi$$

LTL

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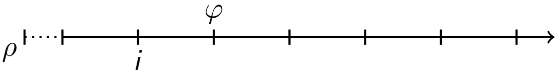
LTL is evaluated at positions i of infinite words ρ over 2^P:

$(\rho, i) \models \mathbf{X}\varphi$:



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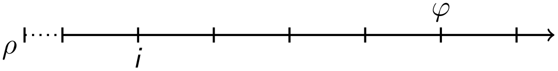
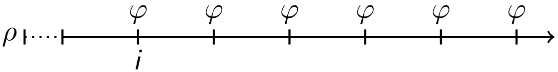


$(\rho, i) \models \mathbf{G}\varphi$:



Parametric LTL

Let $\mathcal{X}$ and $\mathcal{Y}$ be two disjoint sets of variables. PLTL adds bounded temporal operators to LTL:

- $\mathbf{F}_{\leq x}$ for $x \in \mathcal{X}$,
- $\mathbf{G}_{\leq y}$ for $y \in \mathcal{Y}$.

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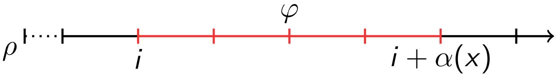
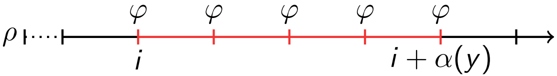
Semantics defined w.r.t. variable valuation $\alpha\colon \mathcal{X} \cup \mathcal{Y} \to \mathbb{N}$.

$(\rho, i, \alpha) \models \mathbf{F}_{\leq x}\varphi$: 

$(\rho, i, \alpha) \models \mathbf{G}_{\leq y}\varphi$: 

Parametric LTL Games

PLTL game $(\mathcal{A}, \varphi)$:

- σ is a winning strategy for Player 0 w.r.t. α iff for all plays ρ consistent with σ: $(\rho, 0, \alpha) \models \varphi$.
- τ is a winning strategy for Player 1 w.r.t. α iff for all plays ρ consistent with τ: $(\rho, 0, \alpha) \not\models \varphi$.

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The set of winning valuations for Player i is

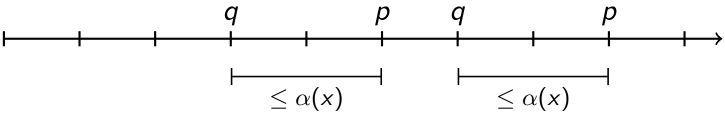
$$\mathcal{W}^i_{\mathcal{G}} = \{\alpha \mid \text{Player } i \text{ has winning strategy for } \mathcal{G} \text{ w.r.t. } \alpha\} \ .$$

We are interested in the emptiness, finiteness, and universality problem for $\mathcal{W}^i_{\mathcal{G}}$ and in finding optimal valuations in $\mathcal{W}^i_{\mathcal{G}}$.

PLTL Games: Example

Winning condition $\mathbf{G}(q \rightarrow \mathbf{F}_{\leq x} p)$: "Every request q is eventually responded by p".

- Player 0's goal: uniformly bound the waiting times between requests q and responses p by $\alpha(x)$.



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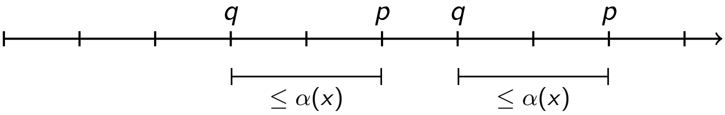
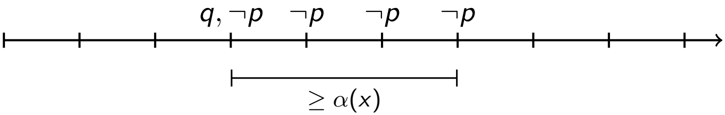


- Player 1's goal: enforce waiting time greater than $\alpha(x)$.



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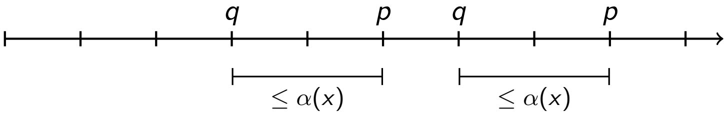

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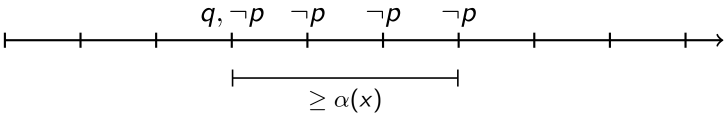

Note: the winning condition induces an optimization problem (for Player 0): minimize $\alpha(x)$.

PLTL: Results

Theorem (Pnueli, Rosner '89)

Determining the winner of an LTL *game is* **2EXPTIME**-*complete.*

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So, adding bounded temporal operators does increase the complexity of solving games.

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If φ contains only $\mathbf{F}_{\leq x}$ respectively only $\mathbf{G}_{\leq y}$, then solving games is an optimization problem: which is the *best* valuation in $\mathcal{W}_{\mathcal{G}}^{0}$?

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Proof idea: obtain (double-exponential) upper bound k on the optimal value by a reduction to an LTL game. Then, perform binary search in the interval $(0, k)$ to find the optimum.

Further research and Open Problems

- Again: tradeoff between size and quality of a finite-state strategy.
- Better algorithms for the optimization problems.
- Hardness results for the optimization problems.

Outline

1. Infinite Games

2. Poset Games

3. Parametric LTL Games

4. **Finite-time Muller Games**

5. Conclusion

Motivation

σ positional strategy: $\sigma(w)$ only depends on the last vertex of w.

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Is there an analogous notion for games with finite-state strategies? Here, we consider Muller games.

Muller Games

- $\mathrm{Inf}(\rho) = \{v \in V \mid \exists^\omega n \in \mathbb{N} \text{ such that } \rho_n = v\}$.

Muller game: $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$ such that $\{\mathcal{F}_0, \mathcal{F}_1\}$ is a partition of $2^V \setminus \{\emptyset\}$. A play ρ is winning for Player i, if $\mathrm{Inf}(\rho) \in \mathcal{F}_i$.

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Theorem

Muller games are determined with finite-state strategies of size $|V| \cdot |V|!$.

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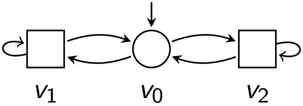

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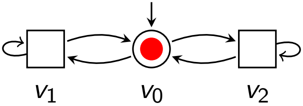

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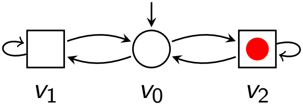

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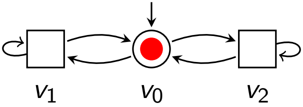

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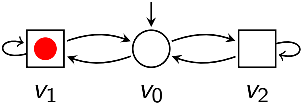

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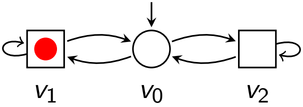

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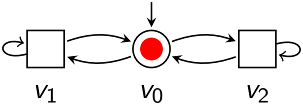

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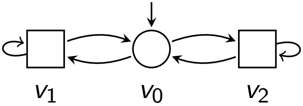

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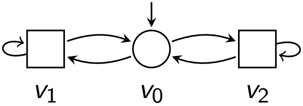

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Theorem

Finite-time Muller games are determined.

First Results

Theorem

Let $\mathcal{A}$ be an arena and $k = |V|^2 \cdot |V|! + 1$. Player i wins the Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$ iff she wins the finite-time Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1, k)$.

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Proof:

A finite-state winning strategy for Player i does not see $F \in \mathcal{F}_{1-i}$ k times in a row.

Further research and Open Problems

Conjecture

Player i wins the Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$ *iff she wins the finite-time Muller game* $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1, 2)$.

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Also:

- Is there a natural definition of eager strategies?
- Complexity of solving a finite-time Muller game? It is just a reachability game (albeit a large one), so simple algorithms exist.
- Starting with a winning strategy for a finite-time Muller game, can we construct a (finite-state) winning strategy for the Muller game.

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Collaboration

Three suggestions from my side:

- Request-response games and Poset games
- PLTL games
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Thank you!