ω-regular and Max-regular Delay Games

Joint work with Felix Klein (Saarland University)

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Introduction

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But assuming fixed interaction might be too strong in the presence of buffers, asynchronous communication channels, etc.

- Hosch & Landweber ('72), Holtmann, Kaiser & Thomas ('10): allow one player to delay her moves, thereby gain a lookahead on her opponents moves.

The Delay Game $\Gamma_f(L)$

- Delay function: $f \colon \mathbb{N} \to \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
- Two players: Input (I) vs. Output (O).

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Questions we are interested in:

- Given L, is there an f such that O wins $\Gamma_f(L)$?
- How *large* does f have to be?
- How hard is the problem to solve?

Examples

- $\binom{\alpha(0)}{\beta(0)}\binom{\alpha(1)}{\beta(1)}\cdots \in L_1 \subseteq (\{a,b\} \times \{a,b\})^\omega$, if $\beta(i) = \alpha(i+2)$.

No delay

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  I wins for every f

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Theorem (Hosch & Landweber '72)

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1. *TFAE for L given by deterministic parity automaton $\mathcal{A}$:*
   - *O wins $\Gamma_f(L)$ for some f.*
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2. *Deciding whether this is the case is in* $2\textsc{ExpTime}$.

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Theorem (Fridman, Löding & Z. '11)

The following problem is undecidable: Given (one-counter, weak, and deterministic) context-free L, does O win $\Gamma_f(L)$ for some f?

Our Results

Theorem (Klein & Z. '14)

1. *TFAE for L given by deterministic parity automaton $\mathcal{A}$ with k colors:*
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2. *Deciding whether this is the case is* ExpTime*-complete.*
3. *Matching lower bound on necessary lookahead (already for reachability and safety).*
4. *Solving reachability delay games is* PSpace*-complete.*

Outline

1. **Reducing Delay Games to Delay-free Games**
2. **Beyond ω-regularity: WMSO+U conditions**
3. **Conclusion**

Upper Bounds for ω-regular Conditions

- Start with deterministic parity automaton $\mathcal{A}$ recognizing the winning condition.
- Extend $\mathcal{A}$ to $\mathcal{C}$ to keep track of maximal color seen during run using states of the form (q, c), which has color c.
- Note: $L(\mathcal{C}) \neq L(\mathcal{A})$.

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- q: state reached by $\mathcal{A}$ after processing $\binom{\alpha(0)}{\beta(0)} \cdots \binom{\alpha(i)}{\beta(i)}$.
- P: set of states reachable by $\mathrm{pr}_0(\mathcal{C})$ from $(q, \Omega(q))$ after processing $\alpha(i+1) \cdots \alpha(j)$.

Proof Continued

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$$r_w^D(q, c) = \delta_{\mathcal{P}}^*(\,\{\,(q, \Omega(q))\,\}\,, w)$$

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- w is witness for $r_w^D \Rightarrow$ Language W_r of witnesses.
- $\mathfrak{R} = \{r \mid W_r \text{ infinite}\}$.

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Lemma

Fix domain D. If $|w| \geq 2^{|\mathcal{C}|^2}$, then w is witness of a unique $r \in \mathfrak{R}$ with domain D.

The Game $\mathcal{G}(\mathcal{A})$

Define new game $\mathcal{G}(\mathcal{A})$ between I and O:

- In round 0:
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Lemma

O wins $\Gamma_f(L(\mathcal{A}))$ for some f if and only if O wins $\mathcal{G}(\mathcal{A})$.

O wins $\Gamma_f(L(\mathcal{A})) \Rightarrow O$ wins $\mathcal{G}(\mathcal{A})$

We can assume f to be constant **[HKT10]**.

$\mathcal{G}$ I: O:

Γ I: O:

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$\mathcal{G}$

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	I:	r_0		r_1
$\mathcal{G}$				
	O:		q_0	

	I:
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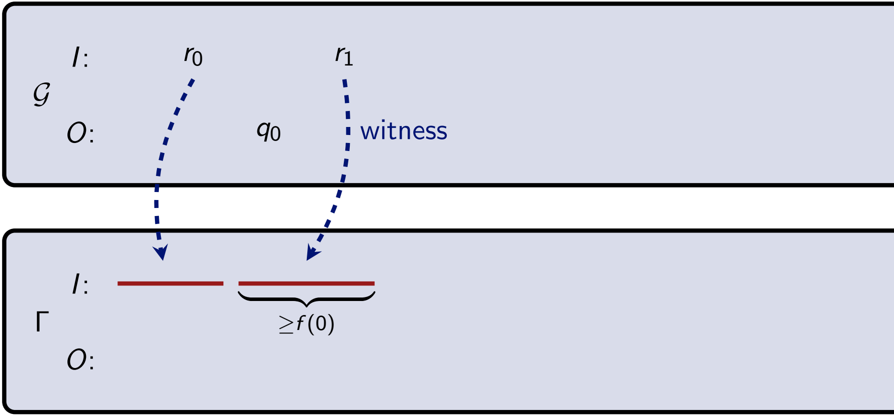

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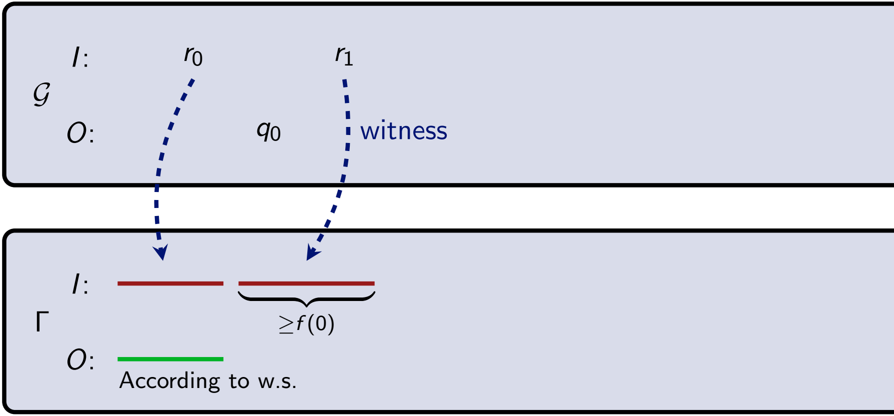

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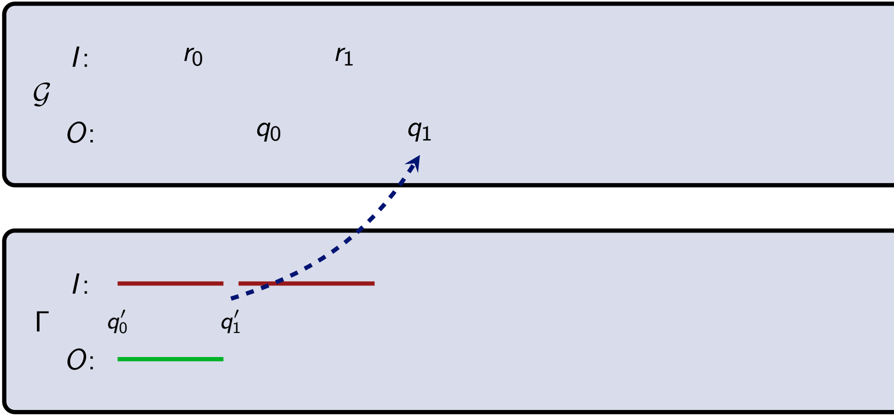

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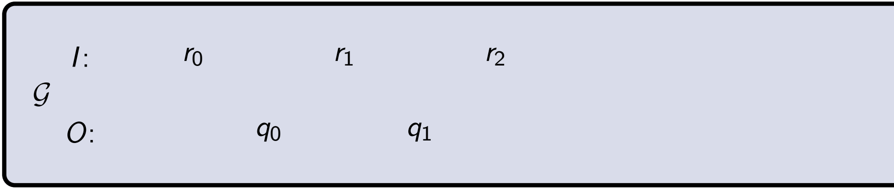

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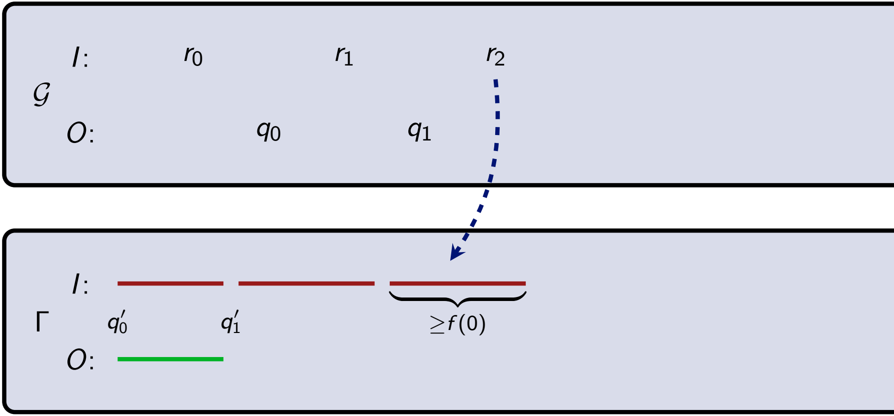

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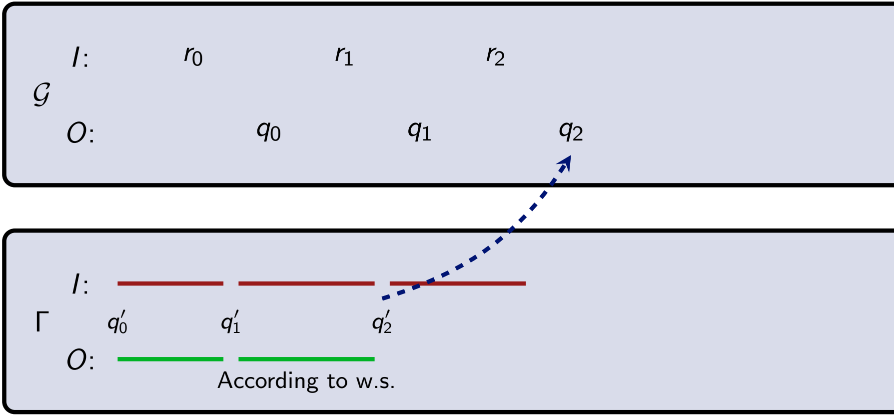

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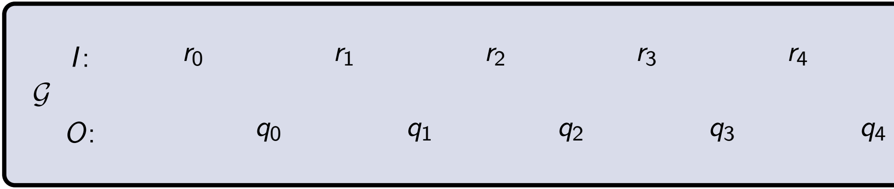

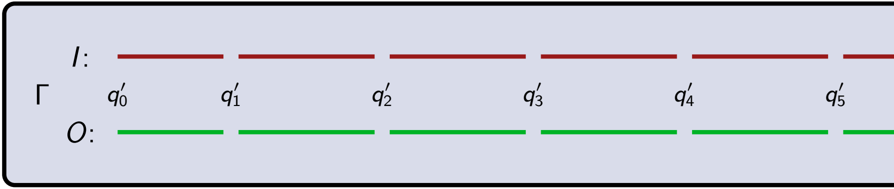

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Let $d = 2^{|\mathcal{C}|^2}$ and $f(0) = 2d$.

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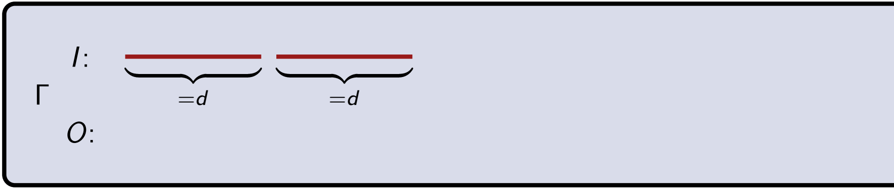

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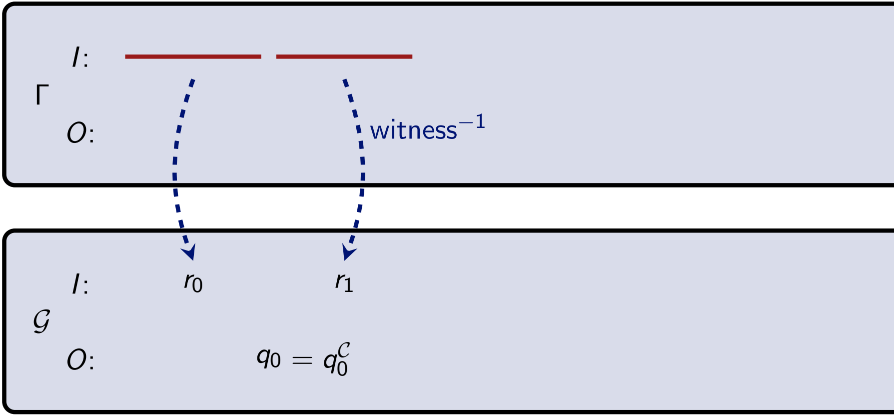

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$\mathcal{G}$

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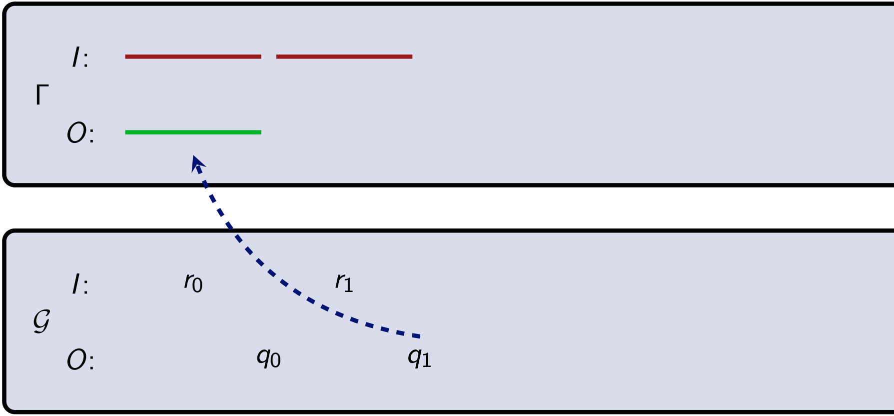

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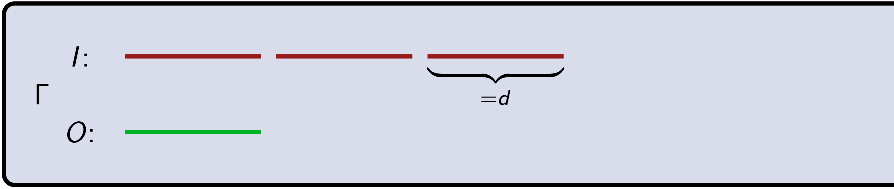

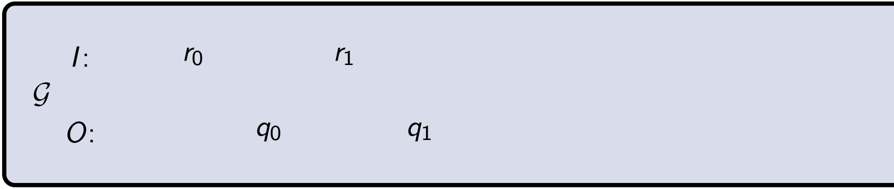

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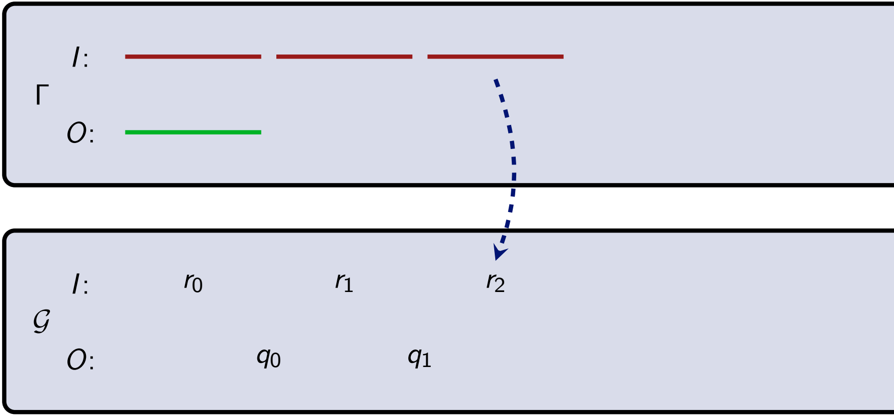

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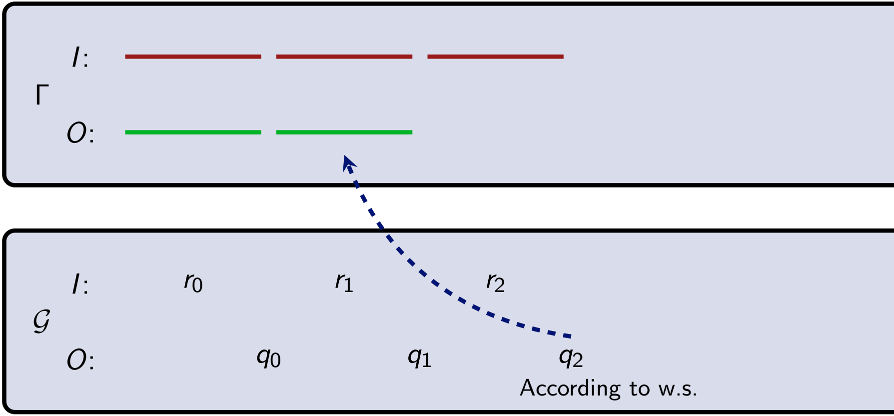

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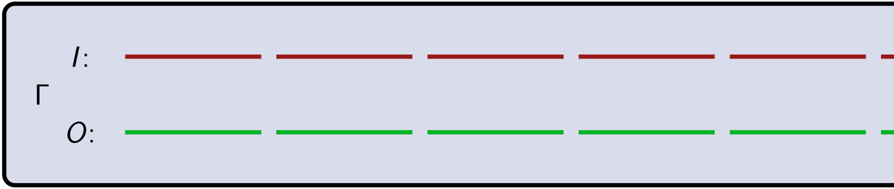

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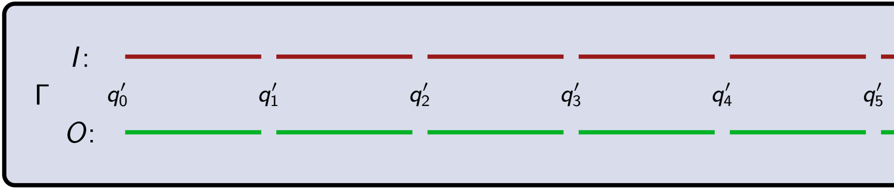

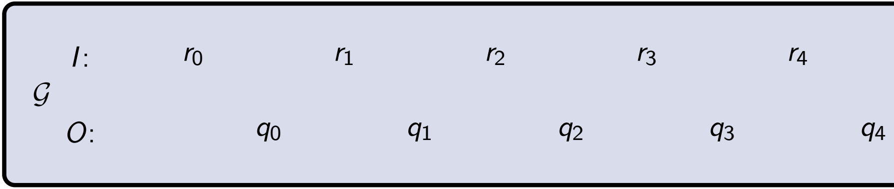

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Corollary

Let $L = L(\mathcal{A})$ where $\mathcal{A}$ is a deterministic parity automaton with k colors. The following are equivalent:

1. *O wins $\Gamma_f(L)$ for some delay function f.*
2. *O wins $\Gamma_f(L)$ for some constant delay function f with $f(0) \leq 2^{(|\mathcal{A}|k)^2+1}$.*

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Note: $f(0) \leq 2^{2|A|k+2} + 2$ achievable by direct pumping argument.

Outline

1. Reducing Delay Games to Delay-free Games

2. **Beyond ω-regularity: WMSO+U conditions**

3. Conclusion

Delay Games with WMSO+U conditions

WMSO+U:

- weak monadic second-order logic with the unbounding quantifier U
- $UX\varphi(X)$: there are arbitrarily large finite sets X s.t. $\varphi(X)$ holds.

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Theorem

The following problem is decidable: Given a max-automaton $\mathcal{A}$, does Player O win $\Gamma_f(L(\mathcal{A}))$ for some constant f?

Proof Sketch

Adapt parity proof: Instead of tracking maximal color, track effect of words over $\Sigma_I \times (\Sigma_O)^*$ on counters:

- Transfers from counter γ to γ'.
- Existence of increments, but not how many.
- $\Rightarrow$ equivalence relation $\equiv$ of exponential index.

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Let $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ be two sequences of words over Σ^* with $\sup_i |x_i| < \infty$, $\sup_i |x'_i| < \infty$, and $x_i \equiv x'_i$ for all i. Then, $x = x_0x_1x_2\cdots \in L(\mathcal{A})$ if and only if $x' = x'_0x'_1x'_2\cdots \in L(\mathcal{A})$.

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- $\mathcal{G}(\mathcal{A})$ is now a game with weak MSO+U winning condition.
- Can be solved as satisfiability problem for weak MSO$+U$ with path quantifiers over infinite tress **[Bojańczyk '14]**.
- Doubly-exponential upper bound on constant delay.

Constant Lookahead is not Sufficient

- $\Sigma_I = \{0, 1, \#\}$ and $\Sigma_O = \{0, 1, *\}$.
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$$\binom{\#}{b}\binom{\alpha(1)}{*}\binom{\alpha(2)}{*}\cdots\binom{\alpha(n)}{*}\binom{b}{b} \in (\Sigma_I \times \Sigma_O)^+$$

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Theorem:
I wins $\Gamma_f(L)$, if f is a bounded delay function, O if f is unbounded.

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Open questions:

- Consider non-deterministic automata and
- Rabin, Streett, Muller automata.
- Can we determine minimal lookahead that is sufficient to win?

Conclusion

Results for max-regular conditions:

- Decidable w.r.t. constant delay functions.
- If O wins w.r.t. some constant delay function, then doubly-exponential constant delay is sufficient.
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Open questions:

- What kind of delay function is sufficient?
- Decidability w.r.t. arbitrary delay functions.