Playing Muller Games in a Hurry

Joint work with John Fearnley, University of Warwick

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Motivation

Robert McNaughton: *Playing Infinite Games in Finite Time.* In: *A Half-Century of Automata Theory*, World Scientific (2000).

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"Winning regions should be equal"

Muller Games

A Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$ consists of an arena $G = (V, V_0, V_1, E)$ and a partition $(\mathcal{F}_0, \mathcal{F}_1)$ of 2^V.

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- Players move a token through the arena ad infinitum.
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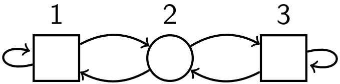

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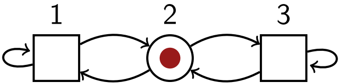

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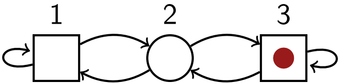

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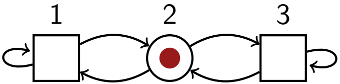

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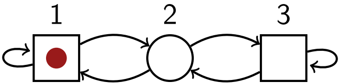

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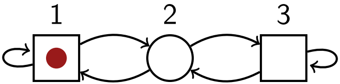

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Winning strategy for Player 0 (circles): coming from 1 to 2 move to 3, coming from 3 to 2 move to 1.

Scoring Functions

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Definition

Finite-time Muller game: $(G, \mathcal{F}_0, \mathcal{F}_1, k)$ with threshold $k \geq 2$.

Rules:

- Players move a token through the arena.
- Stop play w as soon as score of k is reached for the first time.
- There is a unique F such that $\mathrm{Sc}_F(w) = k$ (see above).
- Player i wins w iff $F \in \mathcal{F}_i$.

Results

McNaughton's version: stop play when some Sc_F reaches $|F|! + 1$.

Theorem (McNaughton 2000)

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Stronger statement, which implies the theorem:

Lemma

On her winning region in a Muller game, Player i can prevent her opponent from ever reaching a score of 3 *for every set* $F \in \mathcal{F}_{1-i}$.

Conclusion

Results:

	Reduction	McNaughton	here
Threshold	–	$\lvert F\rvert! + 1$	3
Play Length	$\leq n \cdot n! + 1$	$\leq (n! + 1)^n$	$\leq 3^n$
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Open Questions:

- Is the finite-time Muller game with threshold 2 equivalent to the original Muller game?
- Given a winning strategy for a finite-time Muller game, can we turn it into a winning strategy for the Muller game?