Degrees of Lookahead in Context-free Infinite Games

Joint work with Wladimir Fridman and Christof Löding

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RWTH Aachen University

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Paris, France

Motivation

Starting points:

- Walukiewicz: Solving games with deterministic context-free winning conditions in exponential time.

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- Walukiewicz: Solving games with deterministic context-free winning conditions in exponential time.
- Hosch & Landweber; Holtmann, Kaiser & Thomas: Delay games with regular winning conditions.

Here: delay games with deterministic context-free winning conditions.

- Algorithmic properties.
- Bounds on delay.

Outline

1. **Definitions**
2. Undecidability Results
3. Lower Bounds on Delay
4. Conclusion

The Delay Game $\Gamma_f(L)$

- Delay function: $f : \mathbb{N} \to \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
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I:	0	0	0	0	0	0	0	0	1	0
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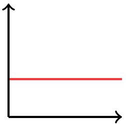
Classifying Delay Functions

1. constant delay function: $f(0) = d$ and $f(i) = 1$ for $i > 0$.

Lookahead
$\sum_{i=0}^{n-1} f(i) - n$

1.)



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Lookahead
$$\sum_{i=0}^{n-1} f(i) - n$$

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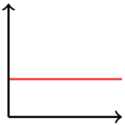

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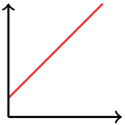

Classifying Delay Functions

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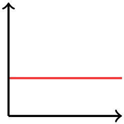
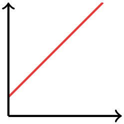
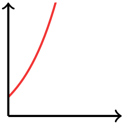
2. linear delay function: $f(i) = k$ for $i \geq 0$.
3. elementary delay function: $[n \mapsto \sum_{i=0}^{n} f(i)] \in \mathcal{O}(\exp_k)$ for some k-fold exponential $\exp_k$.

Lookahead $\sum_{i=0}^{n-1} f(i) - n$

1.) 

2.) 

3.) 

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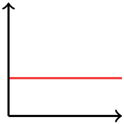
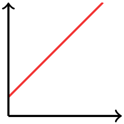
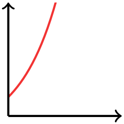
1.) 

2.) 

3.) 

Player O wins the game induced by L with finite (constant, linear, elementary) delay, if there exists an arbitrary (constant, linear, elementary) function f s.t. O has a winning strategy for $\Gamma_f(L)$.

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1. constant delay function: $f(0) = d$ and $f(i) = 1$ for $i > 0$.
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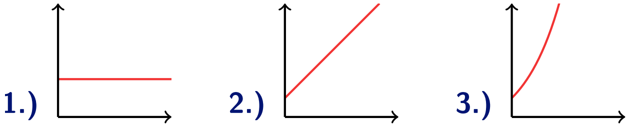
3. elementary delay function: $[n \mapsto \sum_{i=0}^{n} f(i)] \in \mathcal{O}(\exp_k)$ for some k-fold exponential $\exp_k$.

Lookahead $\sum_{i=0}^{n-1} f(i) - n$



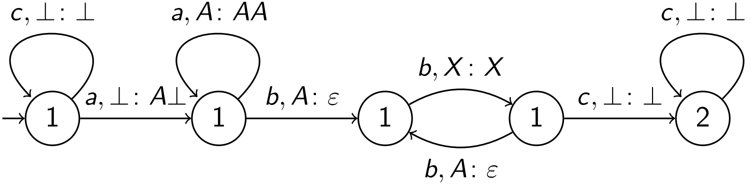
Player O wins the game induced by L with finite (constant, linear, elementary) delay, if there exists an arbitrary (constant, linear, elementary) function f s.t. O has a winning strategy for $\Gamma_f(L)$.

Theorem (HL72, HKT10)

For regular L: Player O wins the game induced by L with finite delay iff she wins it with double-exponential constant delay.

ω-Pushdown Automata

Winning conditions: L recognized by a deterministic ω-pushdown automaton with parity acceptance (parity-DPDA).



Language: $\{c^* a^n b^{2n} c^\omega \mid n > 0\}$.

Outline

1. Definitions
2. **Undecidability Results**
3. Lower Bounds on Delay
4. Conclusion

A Decidable Case

Theorem

The following problem is decidable:

Input: *Parity-DPDA* $\mathcal{A}$ *and* f *s.t.* $\{i \mid f(i) \neq 1\}$ *is finite.*

Question: *Does Player O win* $\Gamma_f(L(\mathcal{A}))$*?*

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Proof Idea

- Suppose $f(0) = 3$, $f(1) = 2$, $f(i) = 1$ for $i > 1$.

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- Suppose $f(0) = 3$, $f(1) = 2$, $f(i) = 1$ for $i > 1$.
- $L' = \{ \binom{\alpha(0)}{\$} \binom{\alpha(1)}{\$} \binom{\alpha(2)}{\beta(0)} \binom{\alpha(3)}{\$} \binom{\alpha(4)}{\beta(1)} \binom{\alpha(5)}{\beta(2)} \cdots \mid \binom{\alpha(0)}{\beta(0)} \binom{\alpha(1)}{\beta(1)} \binom{\alpha(2)}{\beta(2)} \cdots \in L(\mathcal{A}) \}$.
- L' deterministic context-free.

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- L' deterministic context-free.
- Now we have a game without delay.
- Apply Walukiewicz's Theorem: Games with deterministic context-free winning conditions can be solved effectively.

Undecidability

Theorem

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Proof Idea

Preliminaries:

- Reduction from halting problem for 2-register machines.
- Encode configuration (ℓ, n_0, n_1) by $\ell a^{n_0} b^{n_1}$.
- $\ell a^{n_0} b^{n_1} \vdash \ell' a^{n_0'} b^{n_1'}$ is checkable by DPDA.

Proof Idea

- Player I produces configurations c_0, c_1, ...
- Player O can check once whether $c_i \vdash c_{i+1}$ holds.

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Example

```
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```
$   0   $   1   a   $
N
```

```
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Example

\$ 0 \$ 1 a \$ 2 a b \$ 3 a
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Example

\$ 0 \$ 1 a \$ 2 a b \$ 3 a b \$ 4 a b \$
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N	-	N	-	-	N	-	-	-	R_0							

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```

R_0: Player O claims error in `X0`.

Player O wins:
$(3, 1, 1) \nvdash (4, 1, 1)$

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N	-	N	-	-	N	-	-	-	R_0							

- If machine halts, Player I has to cheat. Player O can detect this with linear delay and wins.
- If machine does not halt, Player I can play forever without cheating and wins.

More Undecidability

Corollary

The following problems are undecidable:

- **Input:** *Parity-DPDA $\mathcal{A}$.*
  Question: *Does Player O win the game induced by $L(\mathcal{A})$ with constant delay?*

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The following problems are undecidable:

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Outline

Lower Bounds on Delay

Theorem

There exists a parity-DPDA $\mathcal{A}$ such that Player O wins the game induced by $L(\mathcal{A})$ with finite delay, but for any elementary delay function f, the game $\Gamma_f(L(\mathcal{A}))$ is won by Player I.

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Proof Idea

Adapt idea from undecidability proof:

- Player I produces blocks on which a successor relation is defined (which can be checked by a DPDA).
- Block length grows non-elementary.
- Winning condition forces Player I to cheat at some point.
- Player O wins iff she catches Player I.

Outline

1. Definitions
2. Undecidability Results
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Conclusion

Delay games with context-free winning conditions.

- Determining the winner is undecidable.
- This holds even for visibly one-counter languages accepted by automata with weak acceptance conditions.

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- Again, also for restricted classes of winning conditions.

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Delay games with context-free winning conditions.

- Determining the winner is undecidable.
- This holds even for visibly one-counter languages accepted by automata with weak acceptance conditions.
- Non-elementary lower bounds on delay.
- Again, also for restricted classes of winning conditions.

Open questions:
Undecidability and non-elementary lower bounds, if Player O controls the stack.

- What if Player I controls the stack?
- Linear delay necessary in this case.