Finite-state Strategies in Delay Games

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Motivation

Two goals:

1. Lift the notion of finite-state strategies to delay games.
2. Present uniform framework for solving delay games (which yields finite-state strategies whenever possible).

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Questions:

- What are delay games?
- Why are finite-state strategies important?
- Why do we need a uniform framework?

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In this talk, a game is given by an ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.

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- **Fridman, Löding & Z. ('11)**: Nothing non-trivial is solvable for ω-contextfree delay games, unbounded lookahead necessary.

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All recent (positive) results use variations of the same proof idea.

Finite-state Strategies

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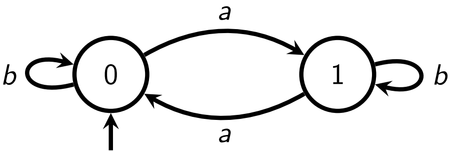
- A finite-state strategy is implemented by a finite automaton with output, and therefore finitely represented.

Example



$w \mapsto |w|_a \bmod 2$

Why Finite-state Strategies

Finite-state/positional strategies are crucial in many applications of infinite games, e.g.:

- In reactive synthesis, a finite-state winning strategy is a correct-by-construction controller.
- (Modern proofs of) Rabin's theorem rely on positional determinacy of parity games.
- In general, the existence of finite-state strategies enables the application of infinite games.
- Determining the memory requirements is one of the most fundamental tasks for a class of games.

Finite-state Strategies for Delay Games

Disclaimer: We focus here on constant delay!

- A strategy in a delay game is still a map $\sigma\colon \Sigma_I^* \to \Sigma_O$.
- So, the classical definition is still applicable.
- By "hardcoding" constant lookahead into the rules of the game, finite-state winning strategies are computable.

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- So, the classical definition is still applicable.
- By "hardcoding" constant lookahead into the rules of the game, finite-state winning strategies are computable.
- However, this notion does not distinguish "past" and "future".

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\begin{array}{lcccccccccccc}
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- Requires 2^d memory states with constant lookahead d.

Block Games

Distinguishing between past and future: block games

- Fix a block length $d > 0$.
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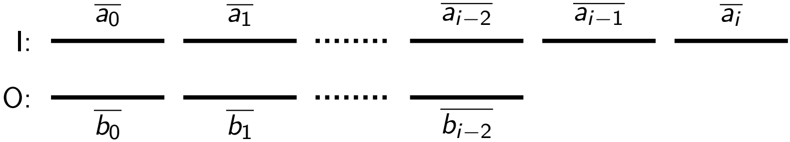

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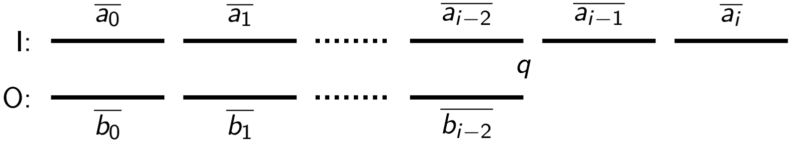

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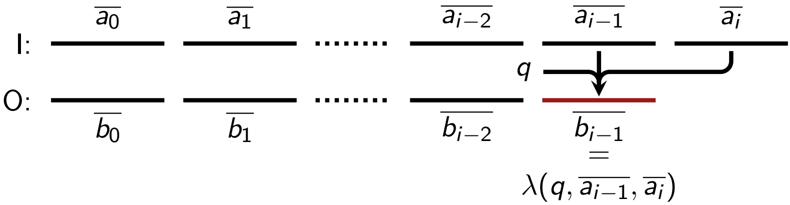

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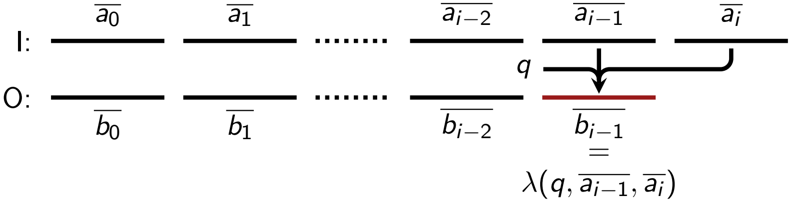

Note:

- Alphabet now exponential in block length!
- But, we distinguish past and future.
- In particular, state complexity only concerned with past.

Aggregations

- Fix ω-automaton $\mathfrak{A}$ and a finite set M.
- $s\colon Q^+ \to M$ is an aggregation for $\mathfrak{A}$, if for all runs $\rho = \pi_0\pi_1\pi_2\cdots$ and $\rho' = \pi_0'\pi_1'\pi_2'\cdots$ with $s(\pi_0)s(\pi_1)s(\pi_2)\cdots = s(\pi_0')s(\pi_1')s(\pi_2')\cdots$: ρ is accepting $\Leftrightarrow$ ρ' is accepting.

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$$\pi_0 \qquad \pi_1 \qquad \pi_2 \qquad \pi_3 \qquad \pi_4 \qquad \cdots$$

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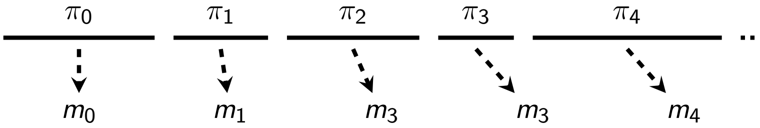

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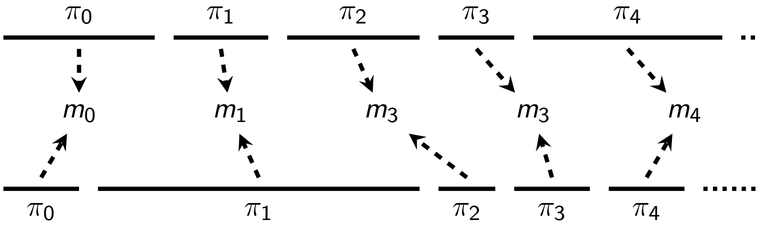

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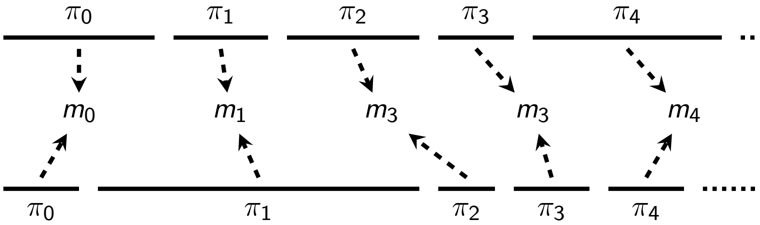

Example

$q_0 \cdots q_i \mapsto \max_{0 \le j \le i} \Omega(q_j)$ is an aggregation for a max-parity automaton with coloring Ω.

Automata Computing Aggregations

Every automaton $\mathfrak{M}$ with input alphabet Q and state set M computes an aggregation $s_{\mathfrak{M}}\colon Q^+ \to M$: $s_{\mathfrak{M}}(\pi)$ is the state reached by $\mathfrak{M}$ when processing π.

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$q_0 \cdots q_i \mapsto \max_{0 \le j \le i} \Omega(q_j)$ computable by automaton with state set $\Omega(Q)$.

Abstract Block Games

Fix $\mathfrak{A}$ recognizing winning condition $L(\mathfrak{A}) \subseteq (\Sigma_I \times \Sigma_O)^\omega$ and let $s_{\mathfrak{M}} \colon Q^+ \to M$ be aggregation for $\mathfrak{A}$ computed by some $\mathfrak{M}$.

- Define $x \equiv x'$ iff x and x' induce the same behavior in $\mathfrak{A}$, i.e., the same state changes and the corresponding runs have the same $s_{\mathfrak{M}}$-value.

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q_0

q_1

$\vdots$

q_n

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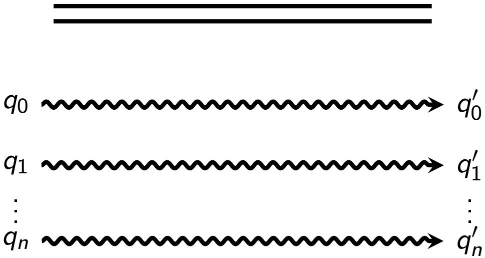

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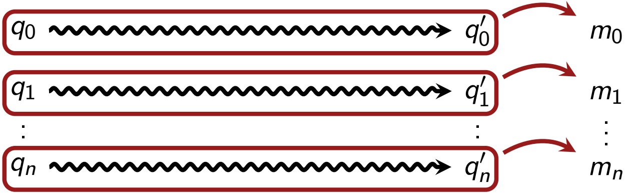

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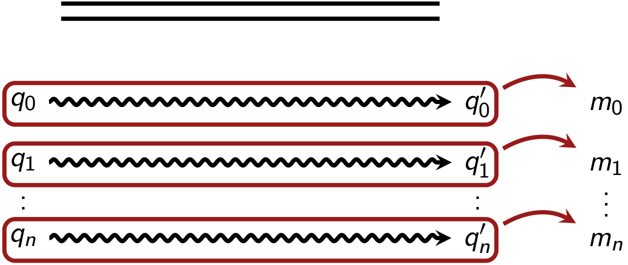

- $\equiv$ has index at most $2^{|Q|^2|M|}$.

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The abstract block game is played as follows:

- Player I picks equivalence classes $S_0 S_1 \cdots$.
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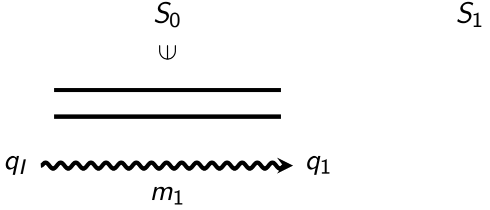

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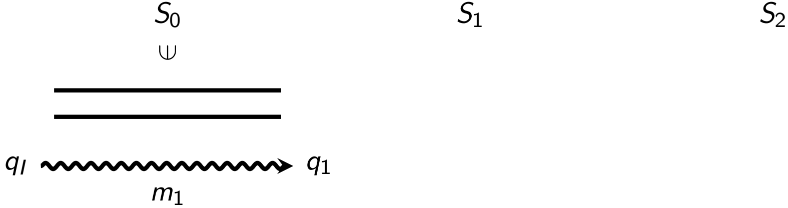

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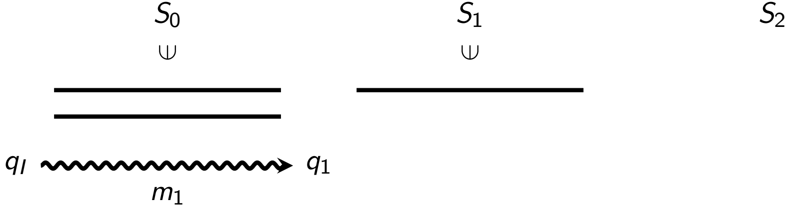

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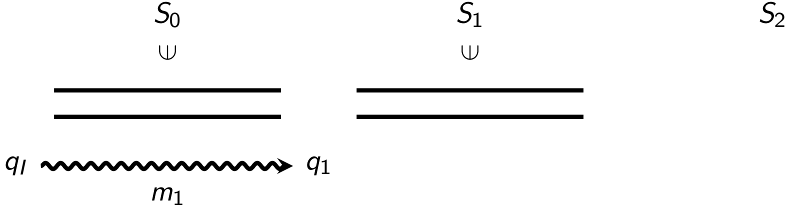

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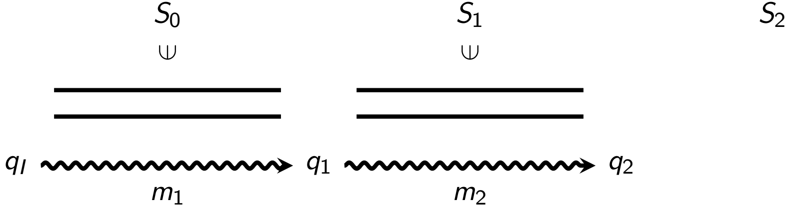

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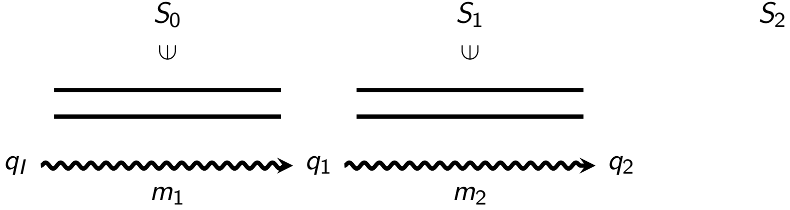

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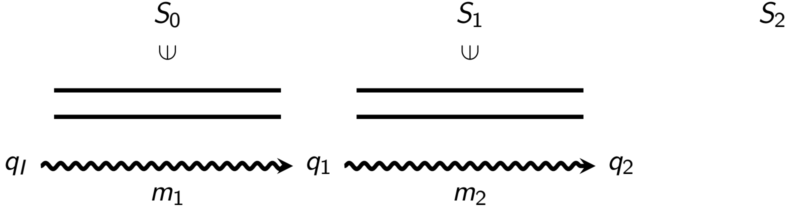

- Player O wins if $m_1 m_2 \cdots$ is aggregation of accepting run.
- This is a delay-free Gale-Stewart game!
- Automaton reconizing winning condition is (roughly) of size $\mathcal{O}(\mathrm{index}(\equiv))$.

Main Theorem

Theorem

Let $\mathfrak{A}$ be an ω-automaton, let $s_{\mathfrak{M}}$ be an aggregation for $\mathfrak{A}$, and define $d = 2^{|Q|^2 \cdot |M|}$.

1. If Player O wins the delay game with winning condition $L(\mathfrak{A})$ for any lookahead, then she also wins the corresponding abstract block game.
2. If Player O wins the abstract block game, then she also wins the block game with winning condition $L(\mathfrak{A})$ and block size d.
3. Moreover, if she has a finite-state winning strategy for the abstract game, then she has a finite-state winning strategy of the same size for the block game.

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Corollary

Solving delay games equivalent to solving abstract block games and constant lookahead $2d$ is sufficient.

Conclusion

Also in the Paper:

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Unpublished (with Sarah Winter):
Recall that automata implementing finite-state strategies in block games process blocks $\Rightarrow$ Exponentially-sized alphabets.

1. Implement transition and output function as transducers.
2. Upper and lower bounds on size in both models.
3. Tradeoffs between these models.