Optimal Strategies in Weighted Limit Games

Joint Work with Aniello Murano (Napoli) and Sasha Rubin (Sydney)
Artwork by Paulina Zimmermann

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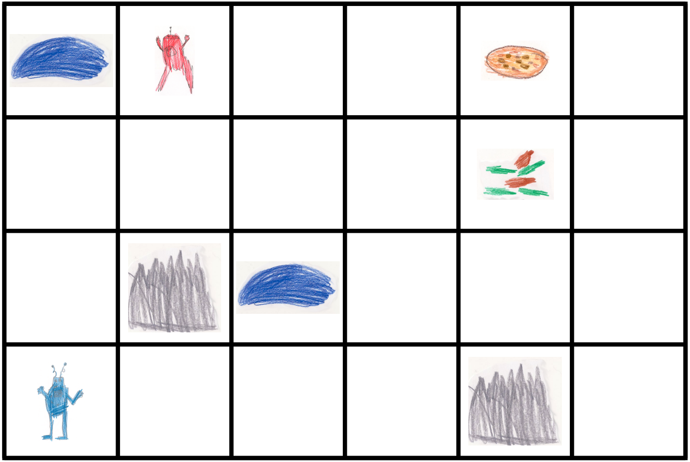

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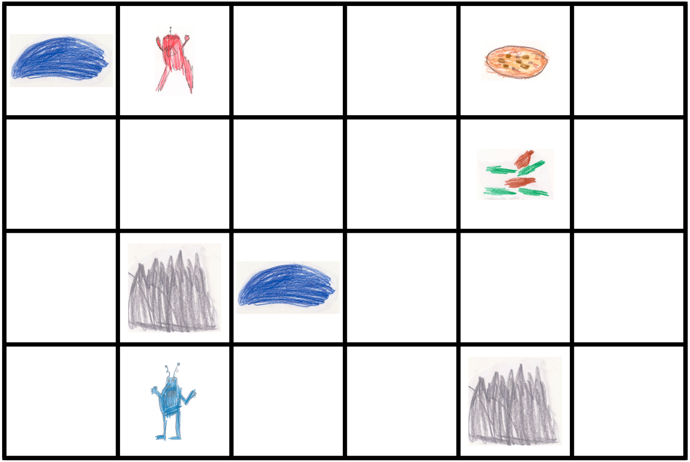

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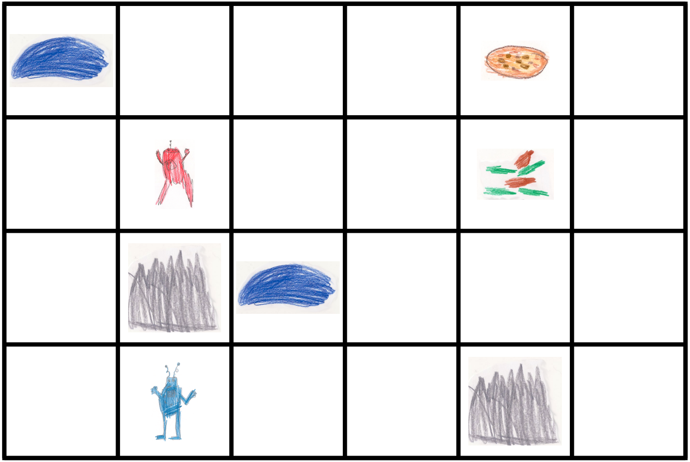

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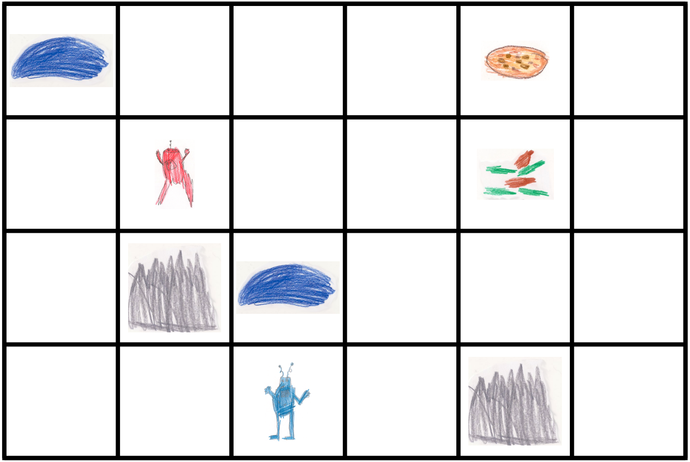

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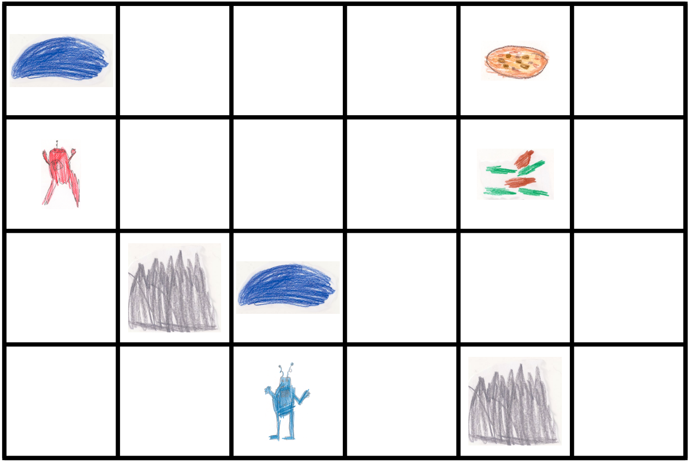

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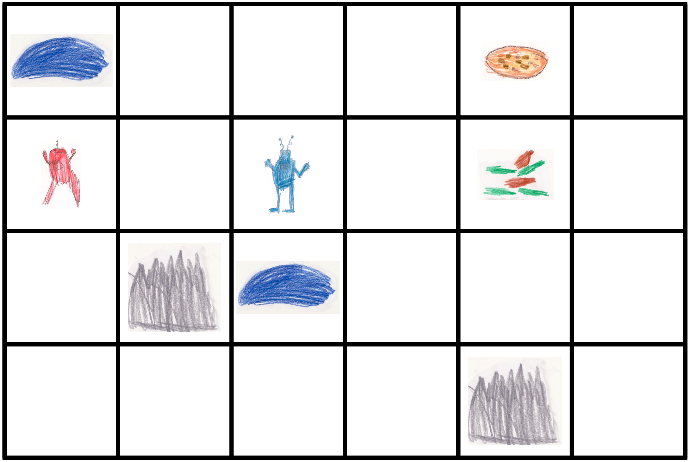

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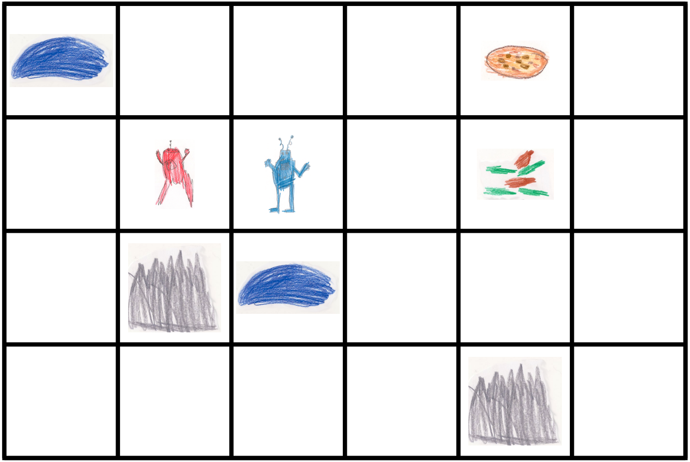

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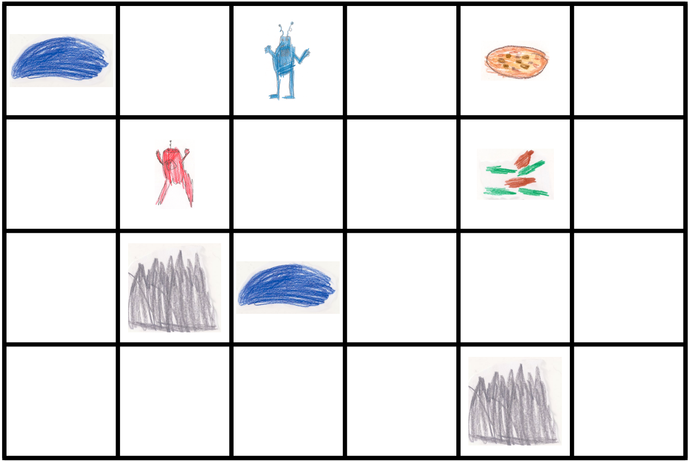

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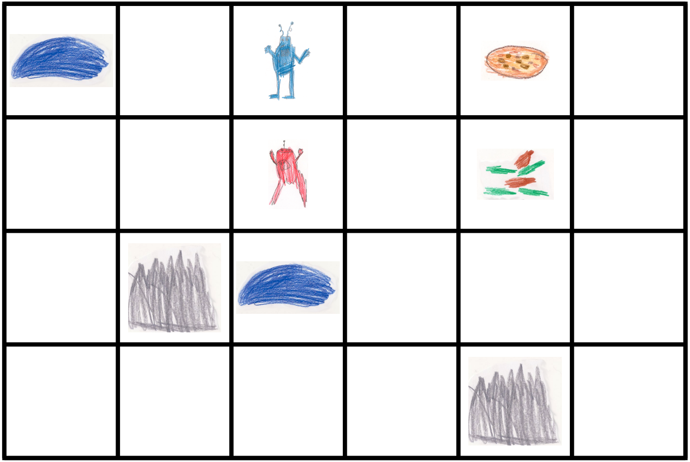

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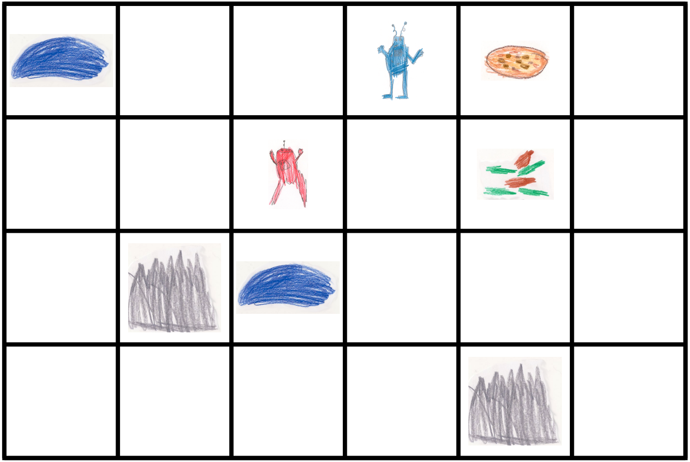

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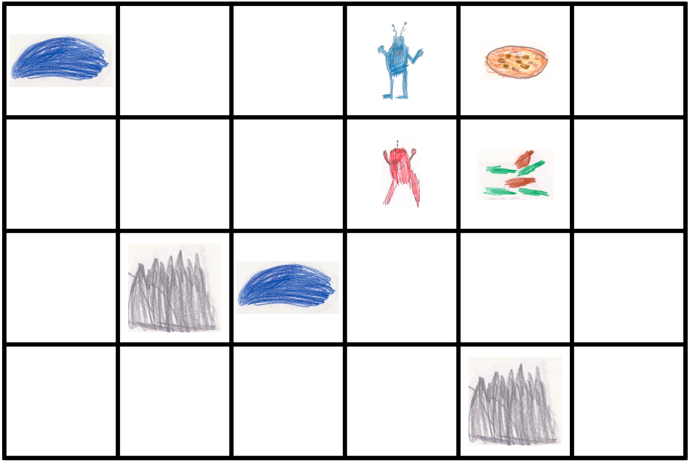

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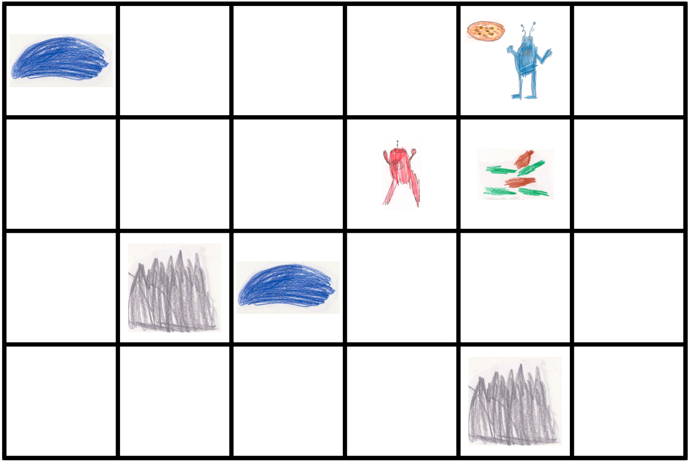

Reactive Synthesis

- The interaction between the robots can be modeled as an infinite-duration two-player game on a graph. The specification yields the winning condition of the game.
- A winning strategy corresponds to an implementation for the blue robot that satisfies the specification.

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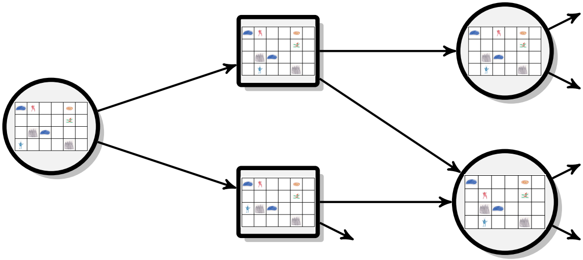

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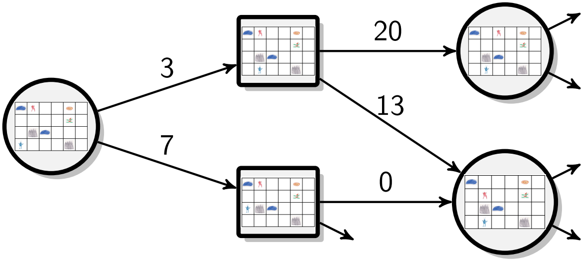

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Reachability specifications:

- Qualitative: reach a fixed set of vertices..
- Quantitative: while minimizing the accumulated weight.
- This problem has been solved before (often as special case of more general problems): optimal strategies exist and can be computed in polynomial time.

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Recurrence specifications:

- Qualitative: reach a fixed set of vertices infinitely often..
- Quantitative: while minimizing the maximal accumulated weight between such visits.
- This problem can also be encoded in more general problems, but a fine-grained analysis is missing

Limit Games

- Limit of a language $K \subseteq V^*$:

  $\lim(K) = \{\alpha_0\alpha_1\alpha_2 \cdots \in V^\omega \mid \alpha_0 \cdots \alpha_j \in K \text{ for inf. many } j\}$.

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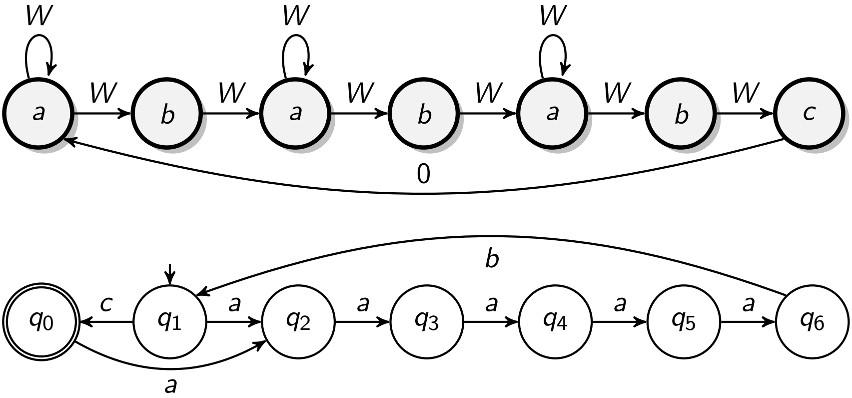
- σ is optimal if $\mathrm{val}_{\mathcal{G}}(\sigma, v) \leq \mathrm{val}_{\mathcal{G}}(\sigma', v)$ for every σ' and every v.

An Example



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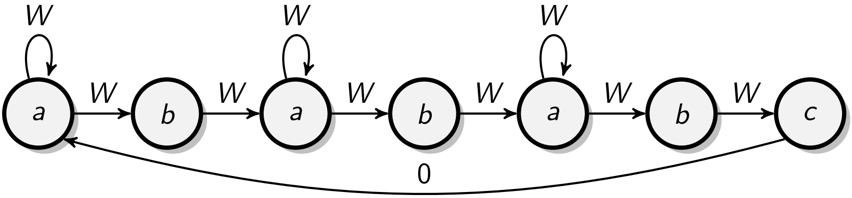

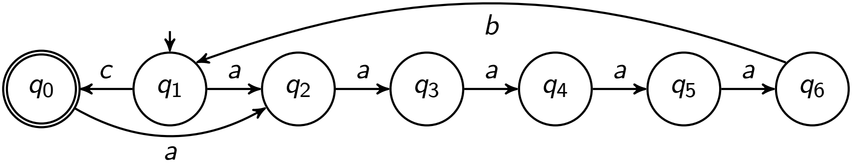

Note

Only Player 0 moves $\Rightarrow$ identify strategies with plays

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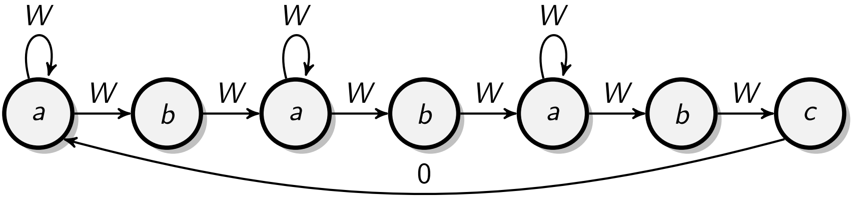

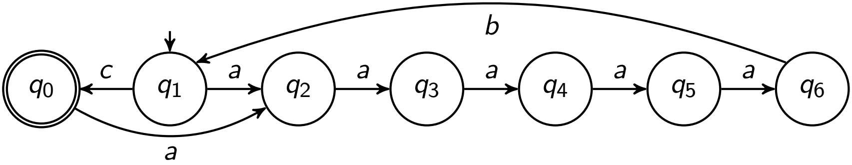

Unique winning play (strategy) of value $18 \cdot W$: every self-loop has to be traversed exactly four times.

A Refinement

Lemma

- $val_{\mathcal{G}}(\rho) < \infty$ *implies* $\rho \in \lim(K)$.
- $val_{\mathcal{G}}(\sigma, v) < \infty$ *implies that* σ *is a winning strategy for Player* 0 *from* v *in* $\mathcal{G}$.

A Refinement

Lemma

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- *$val_{\mathcal{G}}(\sigma, v) < \infty$ implies that σ is a winning strategy for Player 0 from v in $\mathcal{G}$.*

Note

The other directions of both implications can easily be shown to be false. So, “having finite value” is a refinement of “winning”.

$$v_0 \xrightarrow{1} v_1 \xrightarrow{1} v_0 \xrightarrow{1} v_0 \xrightarrow{1} v_1 \xrightarrow{1} v_0 \xrightarrow{1} v_0 \xrightarrow{1} v_0 \xrightarrow{1} v_1 \xrightarrow{1} v_0 \xrightarrow{1} v_0 \xrightarrow{1} v_0 \xrightarrow{1}$$

with $K = (v_0 + v_1)^* v_1$.

Main Results

Theorem

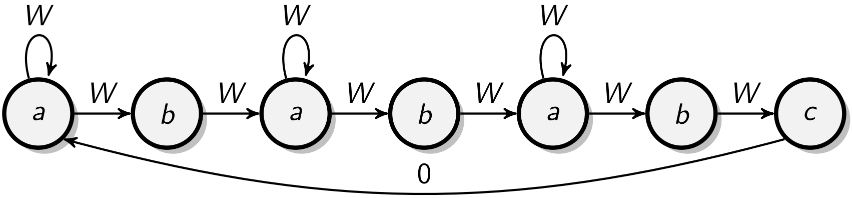
1. *Player 0 has an optimal finite-state strategy in every regular weighted limit game.*

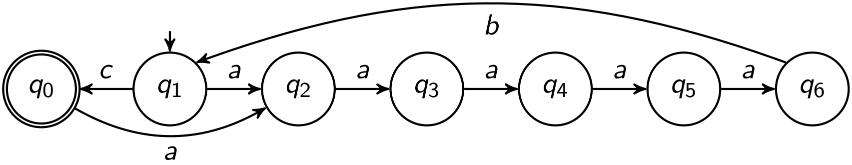
2. *The problem "Given an arena $\mathcal{A}$ and a DFA $\mathfrak{A}$, compute an optimal strategy for Player 0 in $(\mathcal{A}, \lim(L(\mathfrak{A})))$" is solvable in time $\mathcal{O}(|V|^3 \cdot |E| \cdot |Q|^2 \cdot |F|^2)$, where (V, E) is the graph underlying $\mathcal{A}$ and Q and F are the sets of states and accepting states of $\mathfrak{A}$ (using the unit-cost model).*

Upper Bounds

- Value: $(|V| \cdot |Q| + 1) \cdot W$, where W is the largest weight
- Memory size: $|V| \cdot |Q| \cdot |F|$

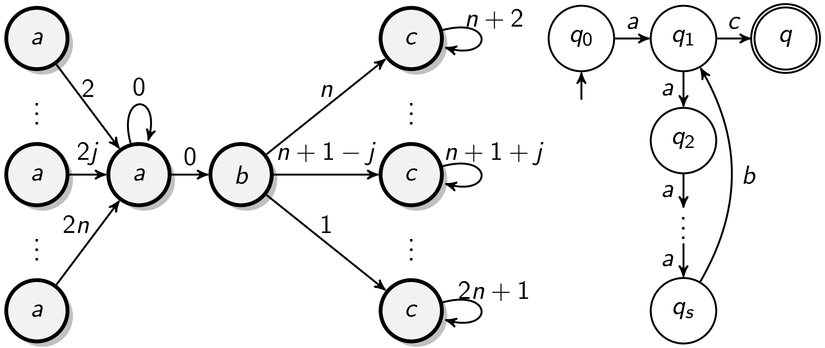
Lower Bounds: Value



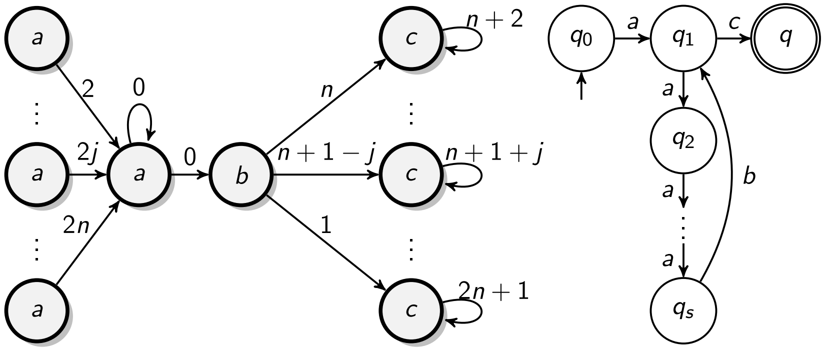


Generalization yields tight lower bound of $|V| \cdot |Q| \cdot W$ on value of optimal strategy.

Lower Bounds: Memory



Lower Bounds: Memory



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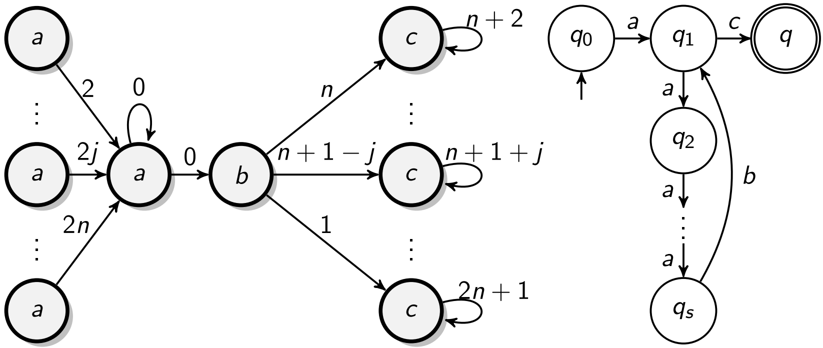
Lower Bounds: Memory



Optimal play from j-th vertex on the left has to use self-loop $s - 2$ times and then reach j-th vertex on the right $\Rightarrow$ requires $n \cdot (s-1)$ memory states.

Thank you for watching.

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