Easy to Win, Hard to Master: Playing Parity Games with Costs Optimally

Joint work with Alexander Weinert (Saarland University)

Martin Zimmermann

Saarland University

September 5th, 2017

University of Liverpool, Liverpool, UK

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Playing … Optimally

Finitary Parity Games

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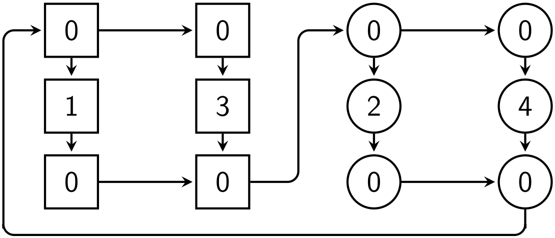
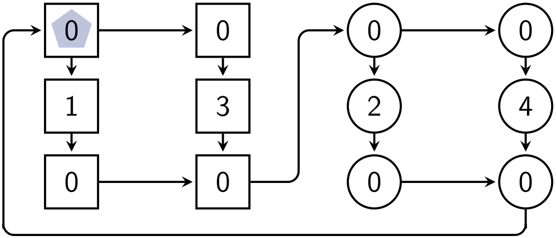
Parity Games



Example due to Chatterjee & Fijalkow

Parity Games



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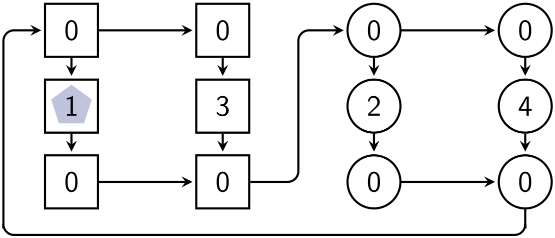
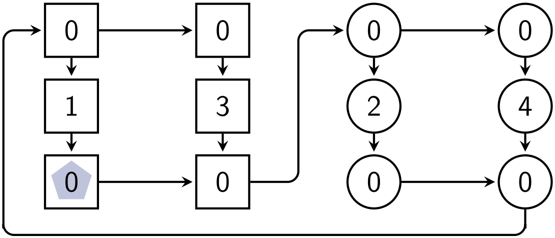


$0 \longrightarrow 1$

Example due to Chatterjee & Fijalkow

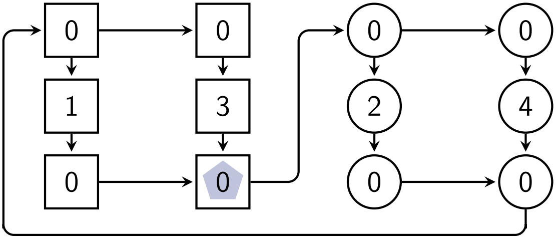
Parity Games



$0 \longrightarrow 1 \longrightarrow 0$

Example due to Chatterjee & Fijalkow

Parity Games



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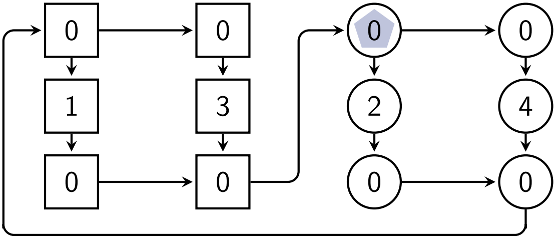

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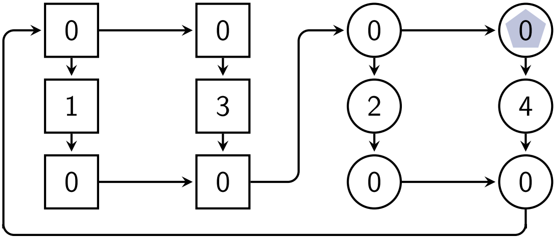

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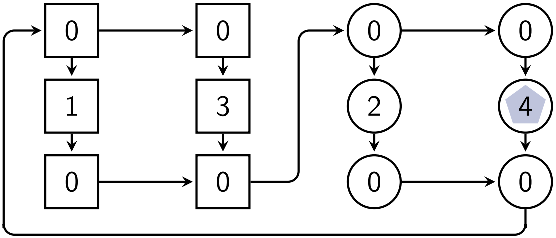

$$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4$$

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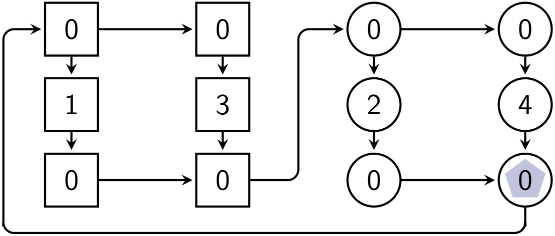

$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4 \longrightarrow 0$

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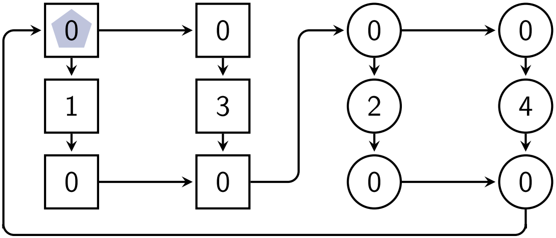

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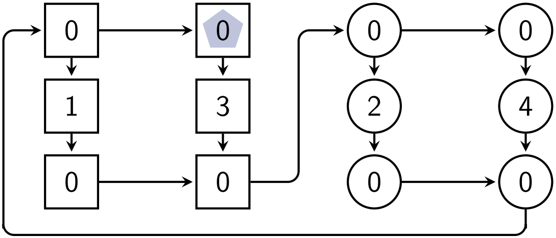

$$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4 \longrightarrow 0$$

$$\hookrightarrow 0 \longrightarrow 0$$

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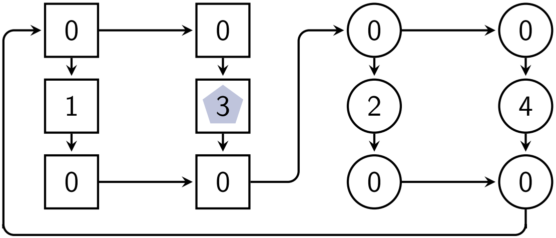
Parity Games



$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4 \longrightarrow 0$

$0 \longrightarrow 0 \longrightarrow 3$

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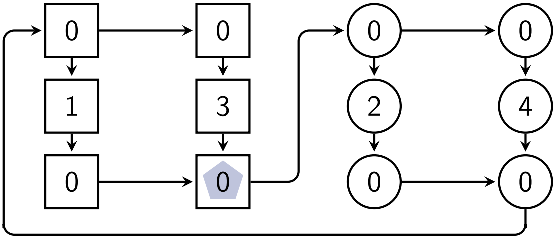

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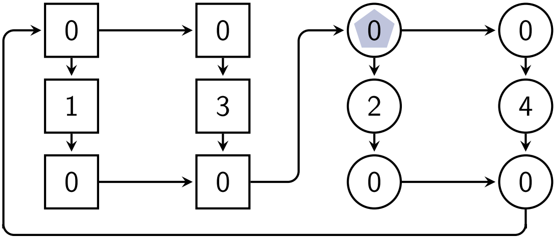

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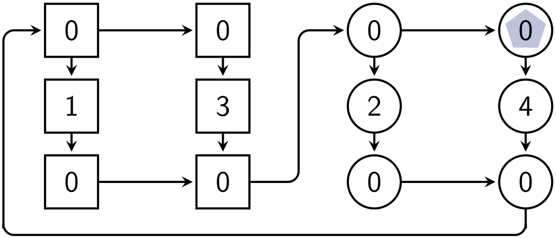

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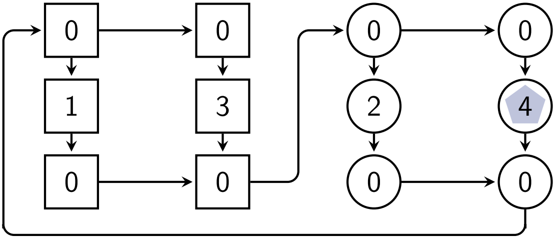

$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \longrightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$

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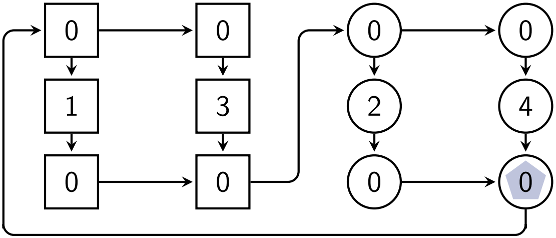

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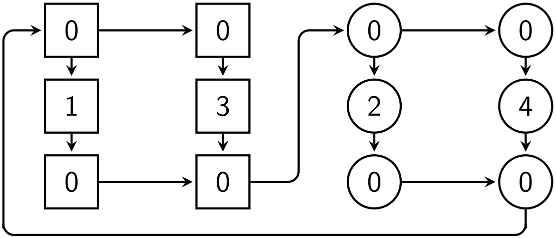

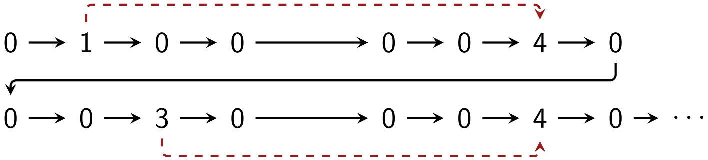

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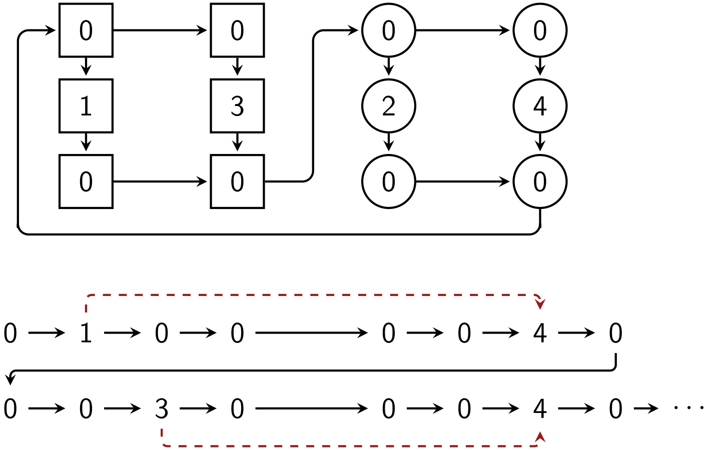
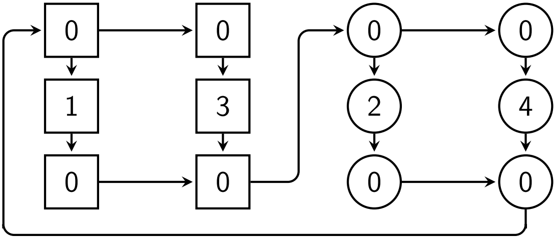


- Various applications: μ-calculus model checking, Rabin's theorem, reactive synthesis, alternating automata,...

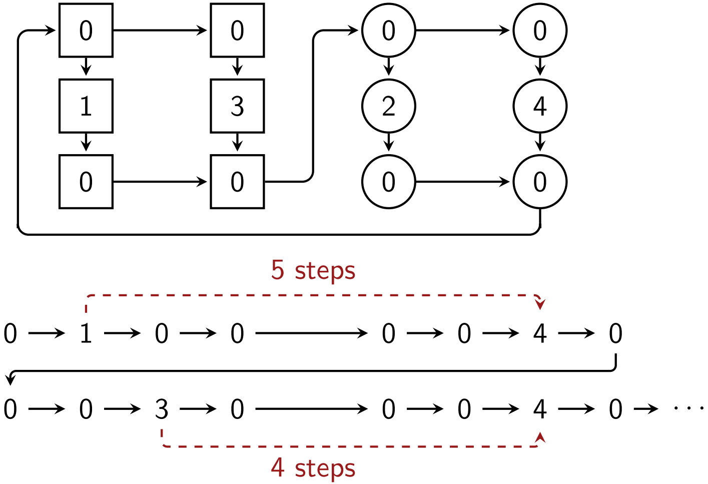
Finitary Parity Games



0 ⟶ 1 ⟶ 0 ⟶ 0 ⟶ 0 ⟶ 0 ⟶ 4 ⟶ 0

0 ⟶ 0 ⟶ 3 ⟶ 0 ⟶ 0 ⟶ 0 ⟶ 4 ⟶ 0 ⟶ $\cdots$

Finitary Parity Games



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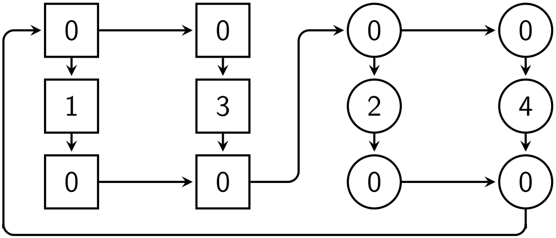
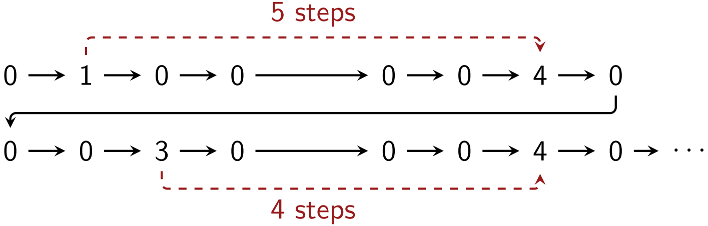
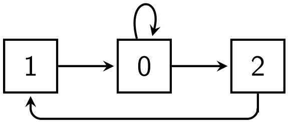




- A quantitative strengthening of parity games.

Another Example



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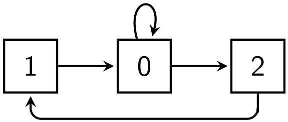

$1 \longrightarrow 0 \longrightarrow 2$

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$1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 2 \rightarrow \cdots$

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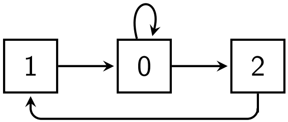

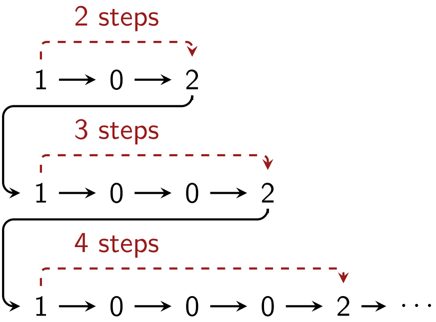

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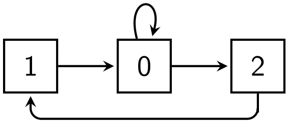

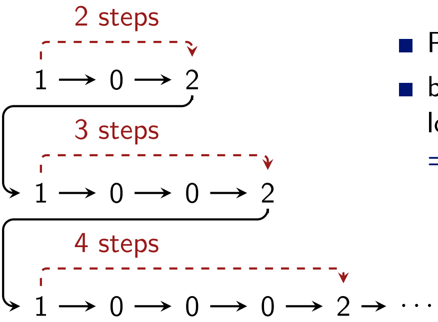

- Player 1 wins from every vertex,
- but needs to stay longer and longer in vertex of color 0.
  ⇒ requires infinite memory.

Previous Work

- **Parity:** Almost all requests are answered.
- **Finitary Parity:** There is a bound b such that almost all requests are answered within b steps.

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Parity	$\mathrm{UP} \cap \mathrm{co\text{-}UP}$	Memoryless	Memoryless
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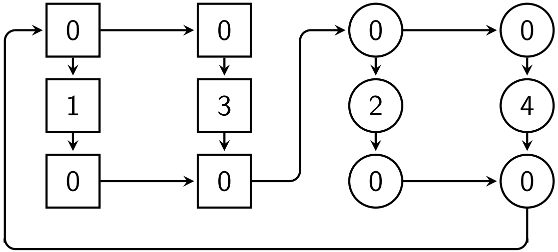
Corollary

If Player 0 wins a finitary parity game $\mathcal{G}$, then a uniform bound $b \leq |\mathcal{G}|$ suffices.

A trivial example shows that the upper bound $|\mathcal{G}|$ is tight.

Back to the Example



Answering requests as soon as possible requires memory.

- Every request can be answered within four steps:
  - a 1 by a 2
  - a 3 by a 4

$\Rightarrow$ requires one bit of memory.

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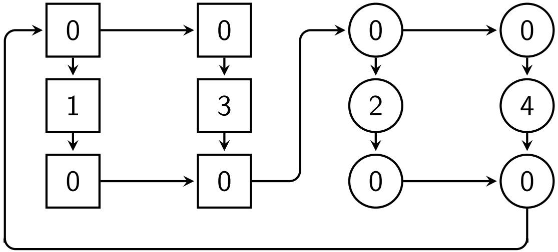

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  $\Rightarrow$ requires one bit of memory.
- But answering a 1 by a 4 takes five steps.

  $\Rightarrow$ every memoryless strategy has at least *cost* 5.

Playing Finitary Parity Games Optimally

Questions

1. How much memory is needed to play finitary parity games optimally?
2. How hard is it to determine the optimal bound b for a finitary parity game?
3. There is a tradeoff between size and cost of strategies! What is its extent?

Outline

1. **Memory Requirements of Optimal Strategies**

2. **Determining Optimal Bounds is Hard**

3. **Trading Memory for Quality and Vice Versa**

4. **Conclusion**

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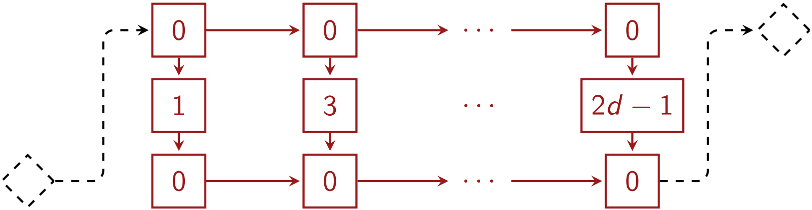
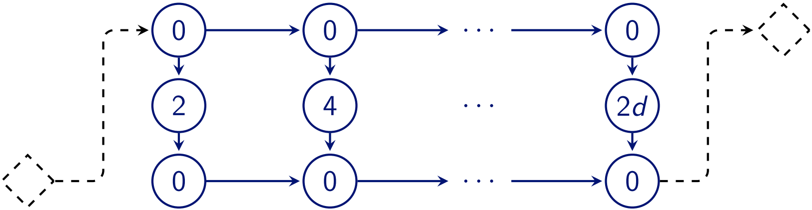
Memory Requirements





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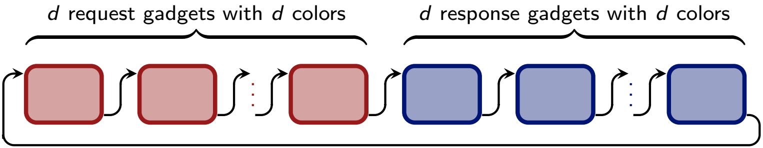
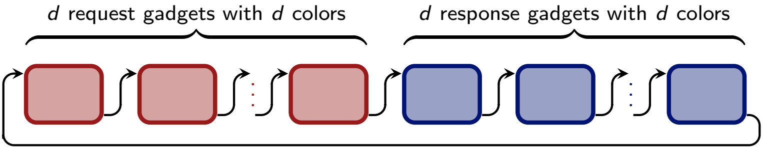


- Player 0 has winning strategy with cost $d^2 + 2d$: answer j-th unique request in j-th response-gadget.
  $\Rightarrow$ requires exponential memory (in d).
- Against a smaller strategy Player 1 can enforce a larger cost, as Player 0 cannot store every sequence of requests.

Memory Requirements



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Theorem

For every $d > 1$, there exists a finitary parity game $\mathcal{G}_d$ such that

- $|\mathcal{G}_d| \in \mathcal{O}(d^2)$ *and $\mathcal{G}_d$ has d odd colors, and*
- *every optimal strategy for Player 0 has at least size 2^{d-1}.*

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1. Memory Requirements of Optimal Strategies
2. **Determining Optimal Bounds is Hard**
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PSPACE-Hardness

Lemma

The following problem is $\textsc{PSpace}$*-hard: "Given a finitary parity game* $\mathcal{G}$ *and a bound* $b \in \mathbb{N}$*, does Player* 0 *have a strategy for* $\mathcal{G}$ *whose cost is at most* b*?"*

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Proof

- By a reduction from QBF (w.l.o.g. in CNF).
- Checking the truth of $\varphi = \forall x \exists y.\ (x \vee \neg y) \wedge (\neg x \vee y)$ as a two-player game (Player 0 wants to prove truth of φ):

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- Checking the truth of $\varphi = \forall x \exists y.\ (x \vee \neg y) \wedge (\neg x \vee y)$ as a two-player game (Player 0 wants to prove truth of φ):
  1. Player 1 picks truth value for x.
  2. Player 0 picks truth value for y.
  3. Player 1 picks clause C.
  4. Player 0 picks literal ℓ from C.
  5. Player 0 wins $\Leftrightarrow$ ℓ is picked to be satisfied in step 1 or 2.

The Reduction

$$\varphi = \forall x \ \exists y \ . \ \overbrace{(\ x \ \vee \ \neg y\ ) \wedge (\ \neg x \ \vee \ y\ )}^{\psi}$$

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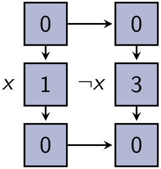

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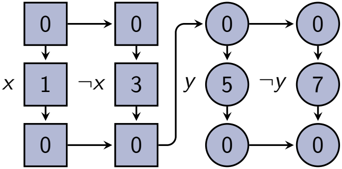

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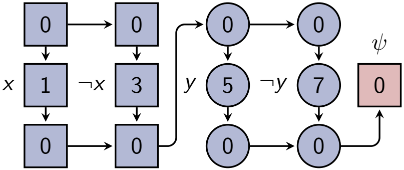

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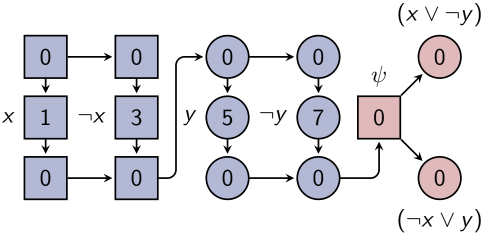

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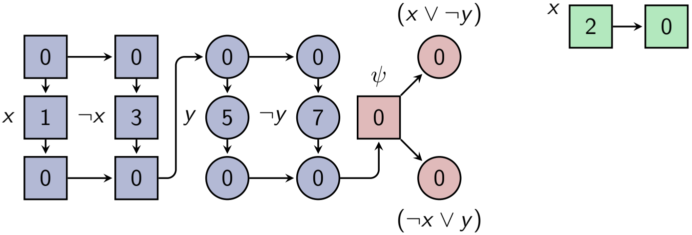

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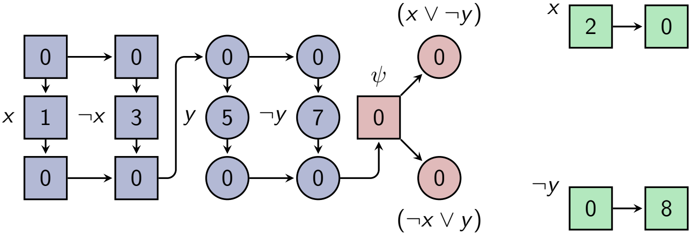

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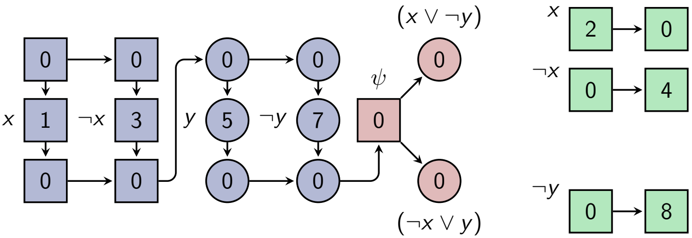

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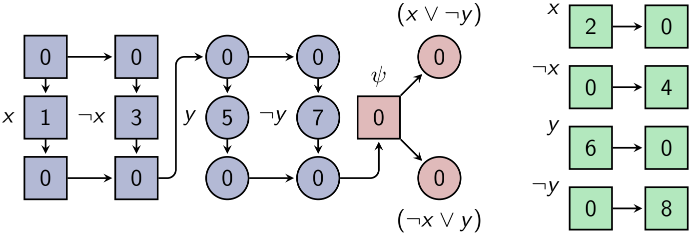

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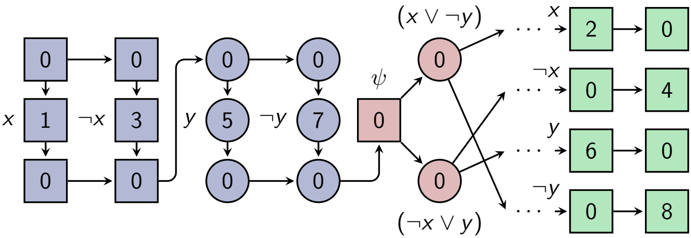

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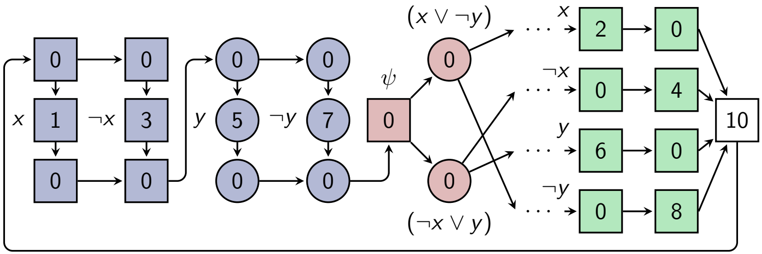

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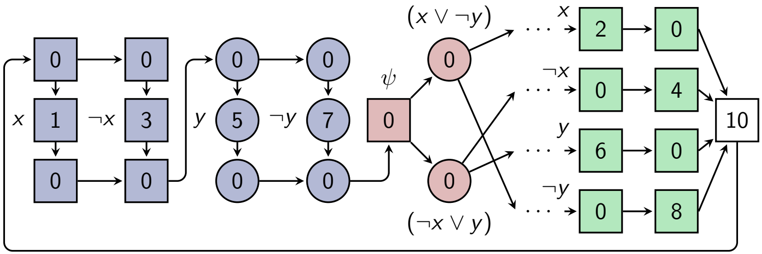

For a well-chosen bound b, a strategy for Player 0 with cost at most b witnesses the truth of φ and vice versa.

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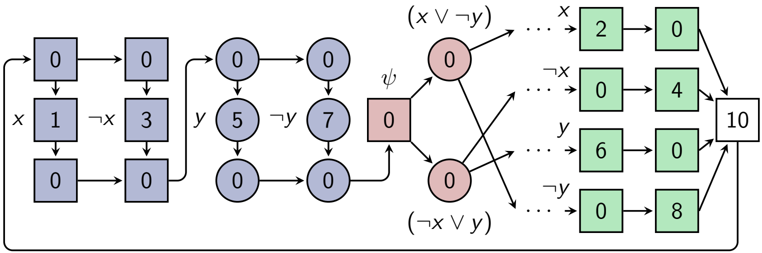
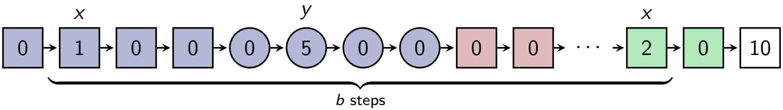


For a well-chosen bound b, a strategy for Player 0 with cost at most b witnesses the truth of φ and vice versa.



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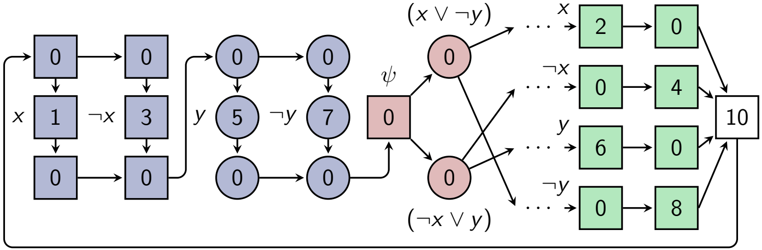

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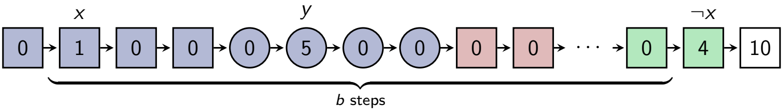

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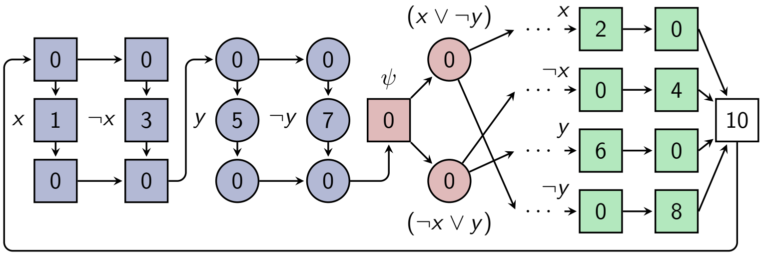

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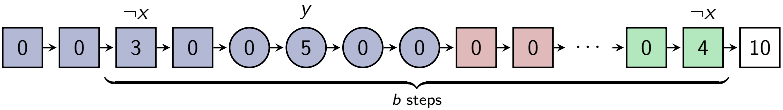

PSPACE-Membership

Lemma

The following problem is in $\textsc{PSpace}$*: "Given a finitary parity game* $\mathcal{G}$ *and a bound* $b \in \mathbb{N}$*, does Player* 0 *have a strategy for* $\mathcal{G}$ *whose cost is at most* b*?"*

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Proof Sketch

Fix $\mathcal{G}$ and b (w.l.o.g. $b \leq |\mathcal{G}|$).

1. Construct equivalent parity game $\mathcal{G}'$ storing the costs of open requests (up to bound b) and the number of overflows (up to bound $|\mathcal{G}|$) $\Rightarrow$ $|\mathcal{G}'| \in |\mathcal{G}|^{\mathcal{O}(d)}$.

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2. Define equivalent finite-duration variant $\mathcal{G}'_f$ of $\mathcal{G}'$ with polynomial play-length.
3. $\mathcal{G}'_f$ can be solved on alternating polynomial-time Turing machine.
4. $\mathrm{APT{\scriptsize IME}} = \mathrm{PS{\scriptsize PACE}}$ concludes the proof.

Upper Bounds on Memory

Equivalence between finitary parity game $\mathcal{G}$ w.r.t. bound b and parity game $\mathcal{G}'$ yields upper bounds on memory requirements.

Corollary

Let $\mathcal{G}$ be a finitary parity game with costs with d odd colors. If Player 0 has a strategy for $\mathcal{G}$ with cost b, then she also has a strategy with cost b and size $(b+2)^d = 2^{d\log(b+2)}$.

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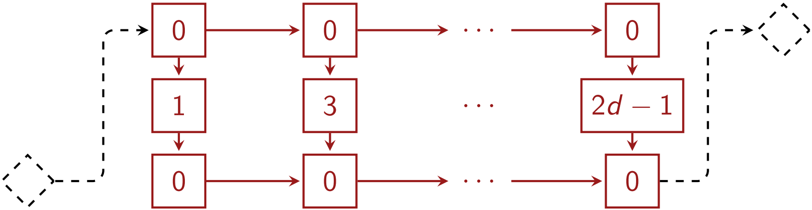
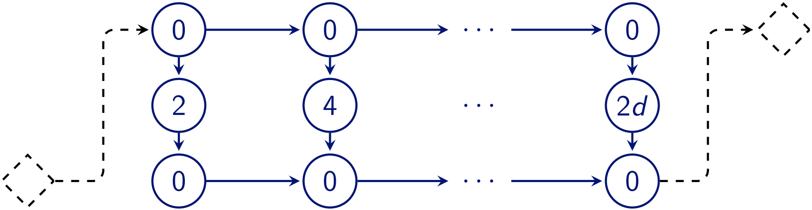
- Recall: lower bound 2^{d-1}.
- The same bounds hold for Player 1.

Outline

1. Memory Requirements of Optimal Strategies
2. Determining Optimal Bounds is Hard
3. **Trading Memory for Quality and Vice Versa**
4. Conclusion

Tradeoffs





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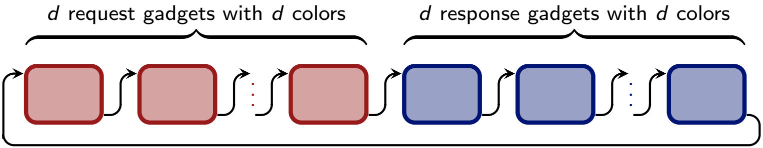

- Recall: Player 0 has winning strategy with cost $d^2 + 2d$: answer j-th unique request in j-th response-gadget, which requires memory of size 2^{d-1}.

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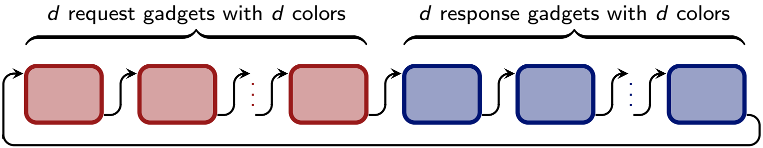
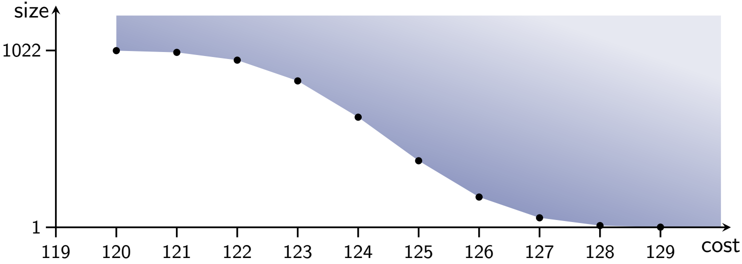


- Recall: Player 0 has winning strategy with cost $d^2 + 2d$: answer j-th unique request in j-th response-gadget, which requires memory of size 2^{d-1}.
- Only store first i unique requests, then go to largest answer in next gadget.
  $\Rightarrow$ achieves cost $d^2 + 3d - i$ and size $\sum_{j=1}^{i-1} \binom{d}{j}$.
- Against a smaller strategy Player 1 can enforce a larger cost, as Player 0 cannot store every sequence of i requests.

Tradeoffs

Theorem

Fix some finitary parity game $\mathcal{G}_d$ as before. For every i with $1 \leq i \leq d$ there exists a strategy σ_i for Player 0 in $\mathcal{G}_d$ such that σ_i has cost $d^2 + 3d - i$ and size $\sum_{j=1}^{i-1} \binom{d}{j}$.
Also, every strategy σ' for Player 0 in $\mathcal{G}_d$ whose cost is at most the cost of σ_i has at least the size of σ_i.



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Conclusion

Results

- Playing finitary games/games with costs optimally is harder than just winning them.
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Open problems

- Parity games with mutiple cost functions
- Multi-dimensional games
- Tradeoffs in other games (first results for parametric LTL and energy games)