Solving Infinite Games with Bounds

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Oberseminar Informatik

Introduction

Verification and synthesis of reactive systems:

- non-terminating,
- dealing with an antagonistic environment.

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- (infinite) game between system and environment,
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Abstract model: graph-based games of infinite duration.

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Abstract model: graph-based games of infinite duration.

Our focus: synthesize not only correct, but optimal systems. Here, optimality depends on context:

- Size of system (memory requirements).
- Response times (quality of the system).
- Generality (allows refinements).

Outline

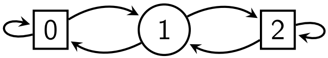
1. Preliminaries
2. Synthesis from Parametric LTL Specifications
3. Playing Muller Games in Finite Time
4. Reductions Down the Borel Hierarchy
5. Conclusion

Outline

Infinite Games

Arena $\mathcal{A} = (V, V_0, V_1, E)$:

- finite directed graph (V, E),
- $V_0 \subseteq V$ positions of Player 0 (circles),
- $V_1 = V \setminus V_0$ positions of Player 1 (squares).



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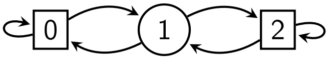

- Play: infinite path $\rho_0\rho_1\cdots$ through $\mathcal{A}$.
- Strategy for Player i: $\sigma\colon V^*V_i \to V$ s.t. $(v, \sigma(wv)) \in E$.
- $\rho_0\rho_1\cdots$ consistent with σ: $\rho_{n+1} = \sigma(\rho_0\cdots\rho_n)$ for all n s.t. $\rho_n \in V_i$.
- Finite-state strategy: implemented by finite automaton with output reading play prefixes.

Infinite Games cont'd

Game: $(\mathcal{A}, \mathrm{Win})$, with $\mathrm{Win} \subseteq V^\omega$ winning plays for Player 0 ($V^\omega \setminus \mathrm{Win}$ winning plays for Player 1).

- Winning strategy σ for Player i from v: every play consistent with σ starting in v is winning for her.
- Winning region of Player i: set of vertices from which she has a winning strategy.

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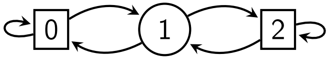
Example



$\mathrm{Win} = \{0, 1\}^\omega$

- Winning region of Player 0: $\{0, 1\}$,
- Winning strategy: from 1 always move to 0.

Outline

PLTL: Syntax and Semantics

Parametric LTL: p atomic proposition, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ ($\mathcal{X} \cap \mathcal{Y} = \emptyset$).

- $\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U} \varphi \mid \varphi \mathbf{R} \varphi \mid \mathbf{F}_{\leq x}\varphi \mid \mathbf{G}_{\leq y}\varphi$

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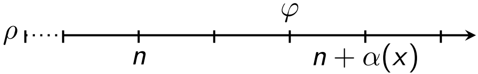
Semantics w.r.t. variable valuation $\alpha\colon \mathcal{X} \cup \mathcal{Y} \to \mathbb{N}$:

- As usual for LTL operators.
- $(\rho, n, \alpha) \models \mathbf{F}_{\leq x}\varphi$:



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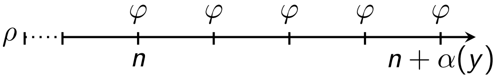

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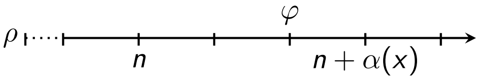

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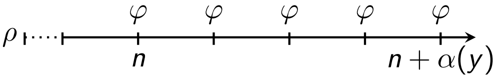

Example:

- Parameterized Büchi: $\mathbf{GF}_{\leq x} p$
- Parameterized request-response: $\mathbf{G}(q \to \mathbf{F}_{\leq x} p)$

PLTL Games

PLTL game: $\mathcal{G} = (\mathcal{A}, v_0, \varphi)$ with arena $\mathcal{A}$ (labeled by $\ell \colon V \to 2^P$), initial vertex v_0, and PLTL formula φ.

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- All plays start in v_0.
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- σ is winning strategy for Player i w.r.t. α, if every consistent play is winning for Player i w.r.t. α.
- Winning valuations for Player i

$$\mathcal{W}_i(\mathcal{G}) = \{\alpha \mid \text{Player } i \text{ has winning strategy for } \mathcal{G} \text{ w.r.t. } \alpha\}$$

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Lemma

Determinacy: $\mathcal{W}_0(\mathcal{G})$ is the complement of $\mathcal{W}_1(\mathcal{G})$.

Decision Problems

- Membership: given $\mathcal{G}$, $i \in \{0, 1\}$, and α, is $\alpha \in \mathcal{W}_i(\mathcal{G})$?
- Emptiness: given $\mathcal{G}$ and $i \in \{0, 1\}$, is $\mathcal{W}_i(\mathcal{G})$ empty?
- Finiteness: given $\mathcal{G}$ and $i \in \{0, 1\}$, is $\mathcal{W}_i(\mathcal{G})$ finite?
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Solving LTL games is $2\textsc{Exptime}$*-complete.*

Adding parameterized operators does not increase complexity:

Theorem (Z. 2011)

All four decision problems are $2\textsc{Exptime}$*-complete.*

Optimization Problems

- $\mathrm{PLTL}_{\mathbf{F}}$: no parameterized always operators $\mathbf{G}_{\leq y}$.
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Let $\mathcal{G}_{\mathbf{F}}$ be a $\text{PLTL}_{\mathbf{F}}$ game with winning condition $\varphi_{\mathbf{F}}$ and let $\mathcal{G}_{\mathbf{G}}$ be a $\text{PLTL}_{\mathbf{G}}$ game with winning condition $\varphi_{\mathbf{G}}$. The following values (and winning strategies realizing them) can be computed in triply-exponential time.

1. $\min_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{F}})} \min_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x)$.

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3. $\max_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{G}})} \max_{y \in \mathrm{var}(\varphi_{\mathbf{G}})} \alpha(y)$.
4. $\max_{\alpha \in \mathcal{W}_0(\mathcal{G}_{\mathbf{G}})} \min_{y \in \mathrm{var}(\varphi_{\mathbf{G}})} \alpha(y)$.

Also: doubly-exponential upper and lower bounds on these values.

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Muller Games

Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$:

- Arena $\mathcal{A}$ and partition $(\mathcal{F}_0, \mathcal{F}_1)$ containing the loops of $\mathcal{A}$.
- Player i wins ρ iff $\mathrm{Inf}(\rho) = \{v \mid \exists^\omega n \text{ s.t. } \rho_n = v\} \in \mathcal{F}_i$.

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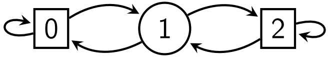
Running example



- $\mathcal{F}_0 = \{\{0, 1, 2\}, \{0\}, \{2\}\}$
- $\mathcal{F}_1 = \{\{0, 1\}, \{1, 2\}\}$

Player 0 has a winning strategy from every vertex: alternate between 0 and 2.

McNaughton's Idea

Robert McNaughton: *Playing Infinite Games in Finite Time.* In: *A Half-Century of Automata Theory*, World Scientific (2000).

> *We believe that infinite games might have an interest for casual living-room recreation.*

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Problem: it takes a long time to play an infinite game! Thus:

- Scoring functions for Muller games.
- Use threshold score to define finite-duration variant.
- **McNaughton 2000**: if threshold is large enough, then the finite-duration game has the same winning regions as the infinite-duration game.

Question

Minimal threshold that guarantees the same winning regions?

Scores and Accumulators

For $F \subseteq V$ define $\mathrm{Sc}_F \colon V^+ \to \mathbb{N}$ and $\mathrm{Acc}_F \colon V^+ \to 2^F$. Intuition:

- $\mathrm{Sc}_F(w)$: maximal $k \in \mathbb{N}$ such that F is visited k times since last vertex in $V \setminus F$ (reset).
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Example:

w	0	0	1	1	0	0	1	2
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$\mathrm{Sc}_{\{0\}}$	1							
$\mathrm{Acc}_{\{0\}}$	$\emptyset$							
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$\mathrm{Acc}_{\{0\}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathrm{Sc}_{\{0,1\}}$	0	0	1	1				
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Scores and Accumulators

For $F \subseteq V$ define $\mathrm{Sc}_F \colon V^+ \to \mathbb{N}$ and $\mathrm{Acc}_F \colon V^+ \to 2^F$. Intuition:

- $\mathrm{Sc}_F(w)$: maximal $k \in \mathbb{N}$ such that F is visited k times since last vertex in $V \setminus F$ (reset).
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Remark

$F = \mathrm{Inf}(\rho) \Leftrightarrow \liminf_{n\to\infty} \mathrm{Sc}_F(\rho_0 \cdots \rho_n) = \infty$.

Finite-Time Muller Games

Two properties of scoring functions:

1. If you play long enough (i.e., $k^{|V|}$ steps), some score will be high (i.e., k).
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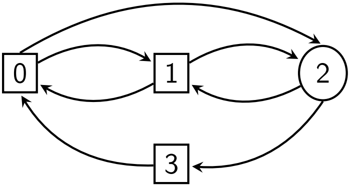
Definition

Finite-time Muller game: $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1, k)$ with threshold $k \geq 2$.

Rules:

- Stop play w as soon as score of k is reached for the first time.
- There is a unique F such that $\mathrm{Sc}_F(w) = k$ (see above).
- Player i wins w iff $F \in \mathcal{F}_i$.

An Example



- $\mathcal{F}_0 = \{\{0,1\},\{1,2\},$
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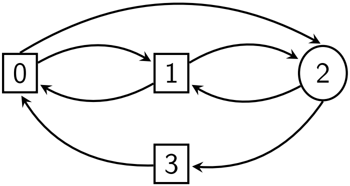

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3

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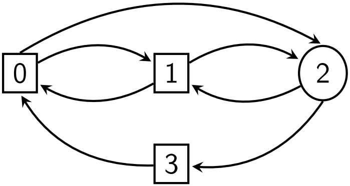

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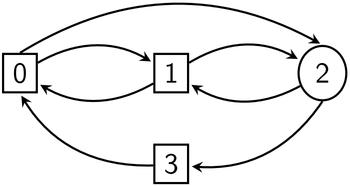

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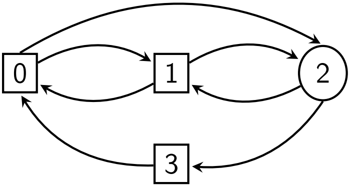

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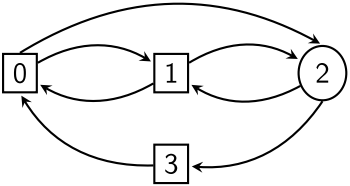

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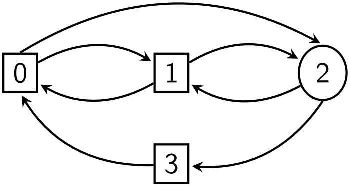

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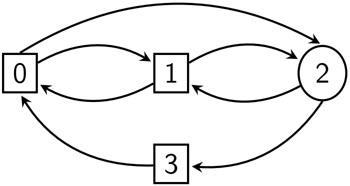

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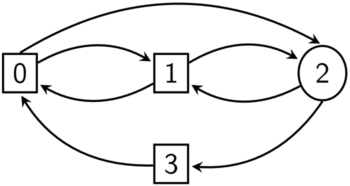

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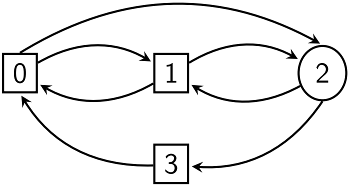

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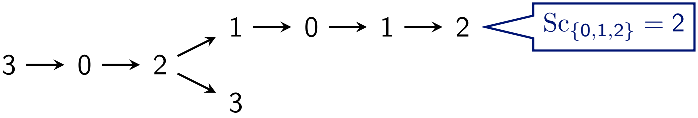

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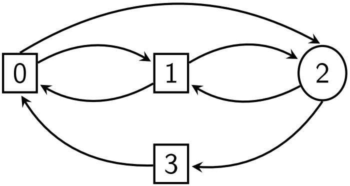

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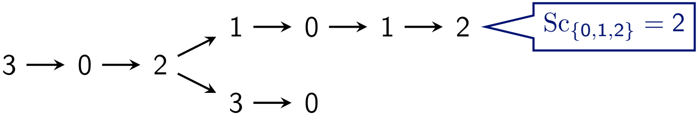

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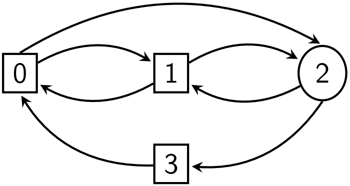

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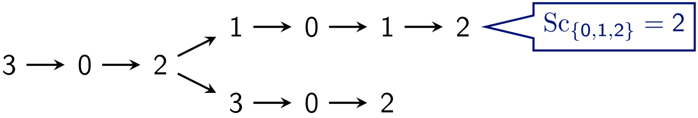

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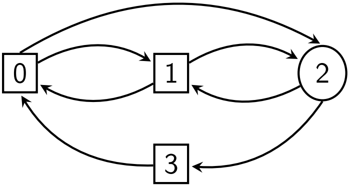

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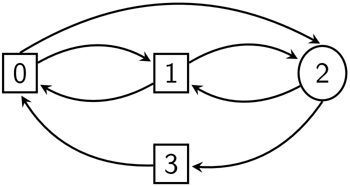

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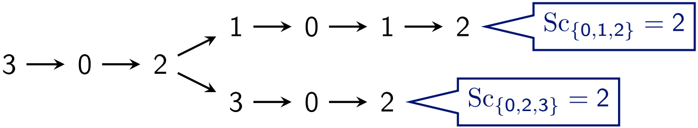

Winning regions are not equal!

Results

McNaughton's version: stop play when some Sc_F reaches $|F|! + 1$.

Theorem (McNaughton 2000)

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Stronger statement, which implies the theorem:

Lemma

On her winning region in a Muller game, Player i can prevent her opponent from ever reaching a score of 3 for some set $F \in \mathcal{F}_{1-i}$.

Comparison of Results

Playing Muller games in finite time in an arena with n vertices:

Variant	Threshold	Maximal play length	
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Teaser (Fridman, Z.)

In pushdown parity games: exponential threshold (stair-) score yields equivalent finite-duration variant.

Outline

Game Reductions

Reduce complicated game $\mathcal{G}$ to simpler game $\mathcal{G}'$: every play in $\mathcal{G}$ is mapped (continuously) to play in $\mathcal{G}'$ that has the same winner.

Solving $\mathcal{G}'$ yields

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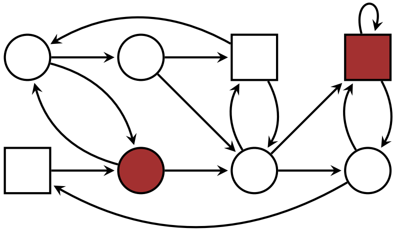

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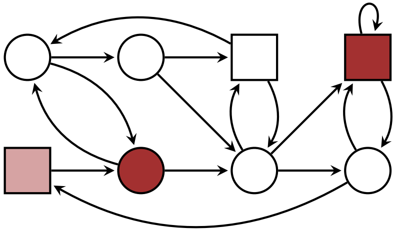

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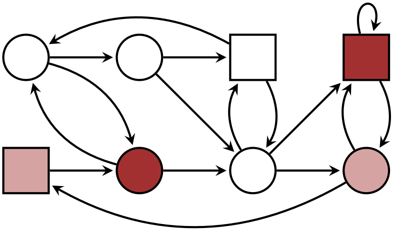

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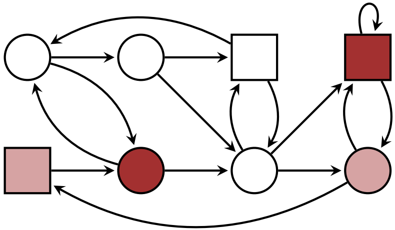

Remark

Muller games cannot be reduced to safety games.

Reducing Muller Games to Safety Games

Lemma

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Thus: v is in Player 0's winning region iff she can prevent Player 1 from reaching a score of 3 starting at v. Safety condition!

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Construction:

- Ignore scores of Player 0.
- Identify plays having the same scores and accumulators for Player 1: $w =_{\mathcal{F}_1} w'$ iff $\mathrm{last}(w) = \mathrm{last}(w')$ and for all $F \in \mathcal{F}_1$:

$$\mathrm{Sc}_F(w) = \mathrm{Sc}_F(w') \text{ and } \mathrm{Acc}_F(w) = \mathrm{Acc}(w')$$

- Build $=_{\mathcal{F}_1}$-quotient of unravelling up to score 3 for Player 1.
- Winning condition for Player 0: avoid $\mathrm{Sc}_F = 3$ for all $F \in \mathcal{F}_1$.

Results

Theorem (Neider, Rabinovich, Z. 2011)

1. *v is in Player i's winning region in the Muller game iff $[v]_{=_{\mathcal{F}_1}}$ is in her winning region in the safety game.*
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Results

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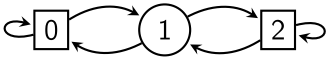
Remarks:

- Size of parity game in LAR-reduction $n!$. But: simpler algorithms for safety games.
- 2. does not hold for Player 1.
- Not a reduction in the classical sense: not every play of the Muller game can be mapped to a play in the safety game.

Proof Idea: Safety to Muller



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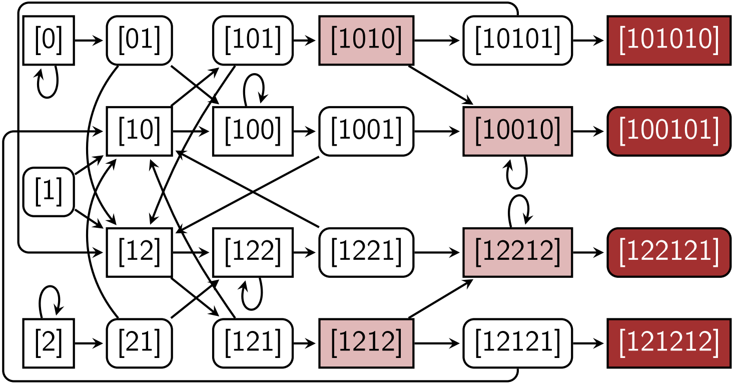

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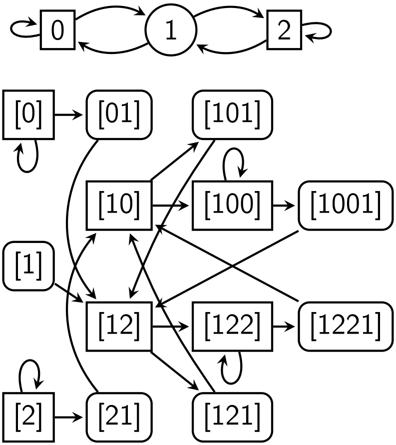

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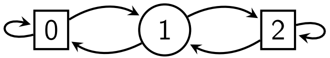

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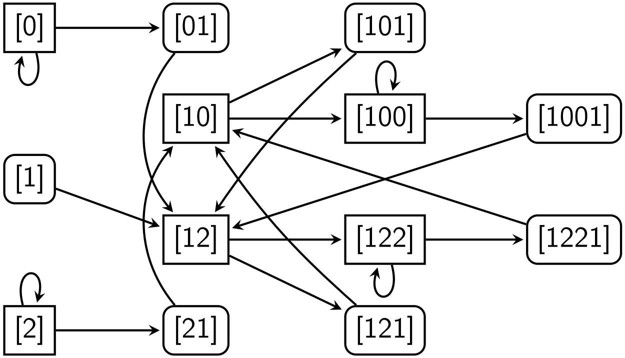

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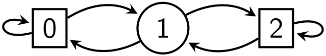

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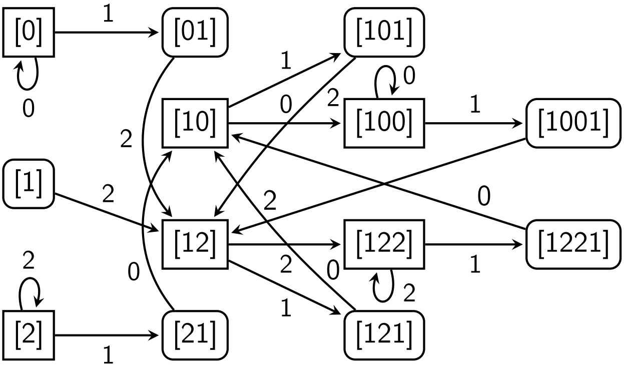

Pick a winning strategy for the safety game. This “is” a finite-state winning strategy for the Muller game.

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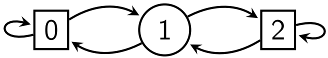

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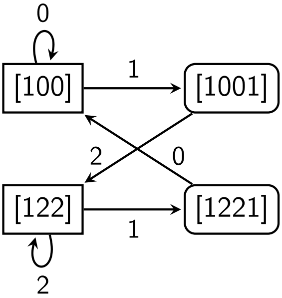

Even better: only use "maximal" elements, yields smaller memory.

Outline

Overview of Results

PLTL games:

- Does Player 0 win w.r.t. some, infinitely many, or all bounds: 2EXPTIME-complete (not harder than solving LTL games).
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Reducing Muller games to safety games:

- Reduce Muller game to safety game of size $(n!)^3$, yields winning regions and one (permissive) winning strategy.
- Generalization for Büchi, co-Büchi, request-response, parity, Rabin, Streett, etc.: yields winning regions and one strategy.

Further Research and Open Questions

PLTL games:

- 2EXPTIME algorithm for optimization problems?
- Tradeoff between size and quality of a finite-state strategy?
- Emptiness problem here: $\exists\sigma\exists\alpha\forall\rho.(\rho,\alpha)\models\varphi$. Non-uniform PLTL games: $\exists\sigma\forall\rho\exists\alpha$ (reminiscent of finitary objectives).

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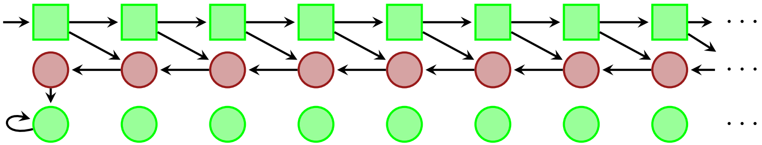
Reducing Muller games to safety games:

- Find "good" winning strategies for safety game $\mathcal{G}_S$ that yield small finite-state winning strategies for Muller game $\mathcal{G}$.
- Progress measure algorithm for Muller games?

Pushdown Arenas

Here: parity games on pushdown arenas (already non-trivial).



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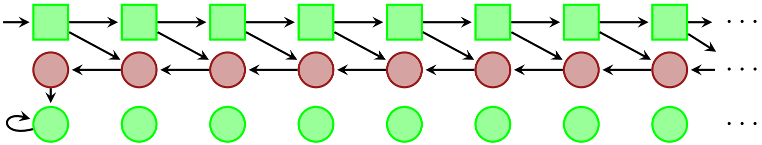

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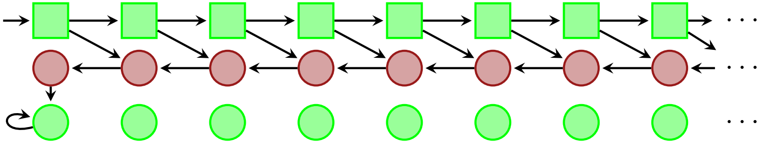

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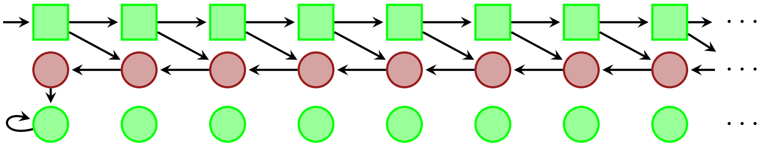

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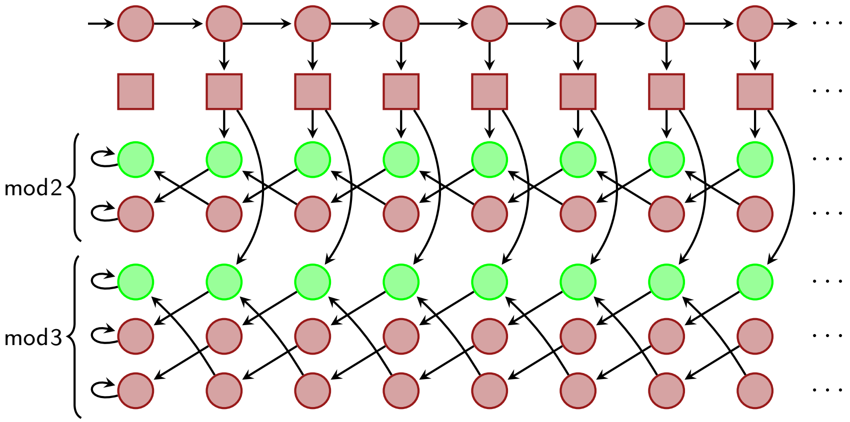
Theorem (Fridman, Z.)

Player i wins $\mathfrak{P}$ iff she wins the finite-duration version.

Pushdown Arenas con't



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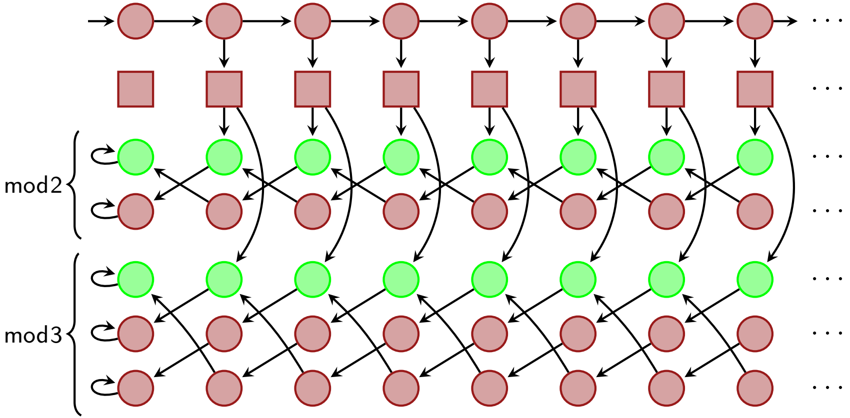

For first n primes $p_1, \ldots, p_n$: Player 0 has to reach stack height $\prod_{j=1}^{n} p_j \approx e^{(1+o(1))n \log n}$ in upper row: cannot prevent losing player from reaching exponentially high scores (in the number of states).

Safety Reductions

Definition

$\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ with vertex set V is safety reducible, if there is a regular $L \subseteq V^*$ such that:

- For every $\rho \in V^\omega$: if $\mathrm{Pref}(\rho) \subseteq L$, then $\rho \in \mathrm{Win}$.
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Theorem (Neider, Rabinovich, Z. 2011)

$\mathcal{G}$ safety reducible with $L(\mathfrak{A}) \subseteq V^*$ for DFA $\mathfrak{A} = (Q, V, q_0, \delta, F)$. Define the safety game $\mathcal{G}_S = (\mathcal{A} \times \mathfrak{A}, V \times F)$. Then:

1. *v is in Player 0's winning region in $\mathcal{G}$ iff $(v, \delta(q_0, v))$ is in her winning region in $\mathcal{G}_S$.*
2. *Player 0 has a finite-state winning strategy for her winning region in $\mathcal{G}$ with memory states Q.*

Safety Reductions: Applications

- Reachability games: reach F after $|V \setminus F|$ steps.
- Büchi games: reach F every $|V \setminus F|$ steps.
- co-Büchi games: avoid visiting $v \in V \setminus F$ twice.
- Request-response games and poset games: bound waiting times (Horn, Thomas, Wallmeier 2008; Zimmermann 2009).
- parity, Rabin, Streett games: progress measure algorithms "are" safety reductions (Jurdziński 2000; Piterman, Pnueli 2006).
- Muller games: bound scores.

If you can solve safety games, you can solve all these games.
Caveat: safety games will be larger than original game.